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Progress in

11

Mathematics

LEARNER'S BOOK



MICHAEL CHIYAKA
FREDERICK FINCH
SYLVIA MULENGA

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How to use this book

Welcome to the *Progress in Mathematics* series for Grades 10–12!

This series is based on the *Senior Secondary Syllabus for Mathematics* issued by the Ministry of Education, Science, Vocational Training and Early Education. All the knowledge, skills and values expressed in the document are addressed in *Progress in Mathematics Grade 11 Learner's Book*, so that you can feel confident about your success in this subject.

This page will help you understand how the book works.

The book is divided into topics so that you can easily see what content will be covered in your Mathematics class.

On the first page of every topic, you will find:

TOPIC 2 Sequences and series

Sub-topic	Specific Outcomes																				
Arithmetic progressions	<ul style="list-style-type: none"> Identify an arithmetic progression (AP) Find the common difference of an arithmetic progression Find the arithmetic mean of an arithmetic progression Find the sum of an arithmetic progression 																				
Geometric progressions	<ul style="list-style-type: none"> Identify a geometric progression (GP) Find the common ratio of a geometric progression Find the common ratio of a geometric progression Find the sum of a geometric progression Find the sum to infinity of a geometric progression 																				
Starter activity	<p>1. Which term of the last three terms in each of the following sequences is the next term? Calculate the sum of the first four terms of the sequence.</p> <p>a) 700, 100, 50, 25, ... b) 2, 5, 12, 19 c) 3, -6, 12, -18</p> <p>2. Below's mother has some cows. She wants to increase the number of cows by two each day. She adds two cows to the herd every day. Her two additions of two cows were on Wednesday the 2nd and Saturday the 5th. You find out that she has 10 cows on Saturday the 5th. How many cows did she have on Wednesday the 2nd?</p> <p>Solution</p> <table border="1"> <tr> <td>Wednesday</td> <td>2</td> <td>3</td> <td>4</td> <td>5</td> </tr> <tr> <td>Thursday</td> <td>3</td> <td>4</td> <td>5</td> <td>6</td> </tr> <tr> <td>Friday</td> <td>4</td> <td>5</td> <td>6</td> <td>7</td> </tr> <tr> <td>Saturday</td> <td>5</td> <td>6</td> <td>7</td> <td>8</td> </tr> </table>	Wednesday	2	3	4	5	Thursday	3	4	5	6	Friday	4	5	6	7	Saturday	5	6	7	8
Wednesday	2	3	4	5																	
Thursday	3	4	5	6																	
Friday	4	5	6	7																	
Saturday	5	6	7	8																	

Topic 2 Sequences and series 11

A table of sub-topics and specific outcomes that will be covered in the topic.

A starter activity helps to introduce the topic with knowledge you already have.

At the end of each topic, you will find the following:

TOPIC 9 Summary, revision and assessment

- Summary**
- Revision**
 - A generalisation is an accurate drawing, using generalised language.
 - The locus of a point is the path along which a point may move in order to satisfy one or more conditions. And is the place of locus.
 - A locus in two dimensions is the path along which a point can move in a two-dimensional space.
 - A locus in three dimensions is the path along which a point can move in three-dimensional space.
- Revision exercises (rwmedin1)**

3.1. Share a line of 9 cm and find the midpoints A and B	(1)
3.2. Rev. 12 pg. 48	(1)
3.3. Construct an angle of 50° and 140°	(1)
3.4. Construct a triangle with sides of 5.2 cm and 5.7 cm	(1)
4.1. Rev. 13 pg. 49	(2 - 3)
4.2. Rev. 14 pg. 49	(2 - 3)
5.1. Construct a polygon with 212 sides of 2.5 cm each	(2)
5.2. Rev. 15 pg. 50	(2)
5.3. Rev. 16 pg. 50	(2)
- Revision exercises**

3.1. Construct a polygon with $PQ = 8\text{ cm}$, $QR = 5\text{ cm}$ and $PR = 3\text{ cm}$	(1)
3.2. Construct a polygon with $AB = 5\text{ cm}$, $BC = 4\text{ cm}$, $CA = 3\text{ cm}$	(1)
4.1. Construct an equilateral triangle with sides of 5.7 cm	(1)
4.2. Rev. 13 pg. 49	(2 - 3)
5.1. Rev. 14 pg. 50	(2)
5.2. Rev. 15 pg. 50	(2)
5.3. Rev. 16 pg. 50	(2)

Topic 9 Summary, revision and assessment 111

The topic summary will help you to revise key learning points in the topic quickly.

Revision exercises help you revise the topic's work and check your understanding.

Assessment exercises help you prepare for tests and exams. _____

Revision and assessment continued

Assessment exercises

1.1	Concentric circles with radii 12, 15, 18, 20 and 22 cm.	(1)
1.2	Two ellipses or ovals. For example: Label the point where the longer axis ends C and D . Label the point where the shorter axis ends E and F .	(1)
1.3	Draw a circle with radius 10 cm and center G .	(1)
1.4	What is the radius of a circle? How is it calculated?	(1)
2.1	Draw a rectangle with a length of 12 cm and a width of 8 cm. In the rectangular space, Points B and D are both equidistant from A and C .	(1)
2.2	Draw the line of symmetry of points B and D .	(1)
2.3	Draw points B and D on different positions than those in lesson, what kind of quadrilaterals are $ABCD$?	(1)
3.1	Draw the line of symmetry of point B which is equidistant from A and C .	(1)
3.2	Construct the line of symmetry of Figure 1.1.	(2)
3.3	Describe the line of symmetry of Figure 1.2. Is it a line from the midpoint of A and C ?	(2)
3.4	Construct the line of symmetry of Figure 1.3.	(2)
3.5	Construct the line of symmetry of Figure 1.4. Is it a line from the midpoint of B and D ?	(2)
3.6	Draw a line in point P which is equidistant from points A and B . The line of point P is such that it always stays from point P .	(1)
4.1	Describe the line of points K and V .	(1)
4.2	If A and B are placed in such a way that the line do not touch at any point, then the line of symmetry of the rectangle between points A and B does not always happen to pass through A and B .	(1)
5.	An elliptical long line passes through our eyes. The focus of point P is such that it is never more than 2 cm from the line.	(1)
5.1	Describe the focus of point P .	(1)
5.2	Describe the focus of point Q .	(1)
6.	A 45° reflection is placed on a straight line AB 30 cm from the vertex of the angle of the shears. In Figure 1.5, describe the locus of the vertex of the angle.	(1)
7.	A long high ladder is placed upright against a wall. The ladder is 10 m long. The distance from the wall to the ladder is 8 m. At the top of the ladder touches the wall, throughout this process, Describe the locus of the midpoint of the ladder.	(4)



Total marks: 50

You will see the following throughout the book:

New words boxes give you the definitions of key words or explain what a certain new word means. These words and the definitions are also in the glossary at the back of the book.

Did you know? boxes give you more and new knowledge about what you are learning.

Worked examples give an example with a model answer that shows step by step how to do a calculation.

Partial variation

If you're partially paid, then the starting salary is s and the pay can be expressed as $y = mx + s$, where m is the rate that part of the partial variation and s is the variable part of the partial variation.

The graph of a partial variation is a straight line that does not pass through the origin.

Worked example 3

The cost of using a car is a form of partial variation, the graph diagram.

1. Determine the equation for the cost of using a car in the first hour.
2. What is the cost of using a car for 2 hours?

Answer:

1. $y = mx + s$
 $y = 2000x + 2000$ (the y-intercept)
 From Topic 3, we know that $s = 2000$
 $y = 2000x + 2000$

2. $y = 2000x + 2000$
 $y = 2000 \times 2 + 2000 = 2000 + 2000 = 4000$
 So, it would cost R2 000 to rent a car for 2 hours.

Activity 4

1. Refer back to the previous example. If the China family paid R10 000 (R1000) for the car, then how long did they own it?
2. The equation of a partial variation is $y = mx + s$. When $x = 15$, $y = 1500$. What is the value of m ?
3. The following points lie on the graph of a partial variation. Determine the equation of the partial variation.
 - 4. If the point (1, 2) lies on the same graph, calculate the value of s .
 - 5. Reproduce the equation $y = mx + s$ for the equation of a partial variation.
 - 6. The equation of a partial variation is $y = mx + s$, where $s = 4$. Family A takes 2 hours to travel 100 km. Determine the value of m .

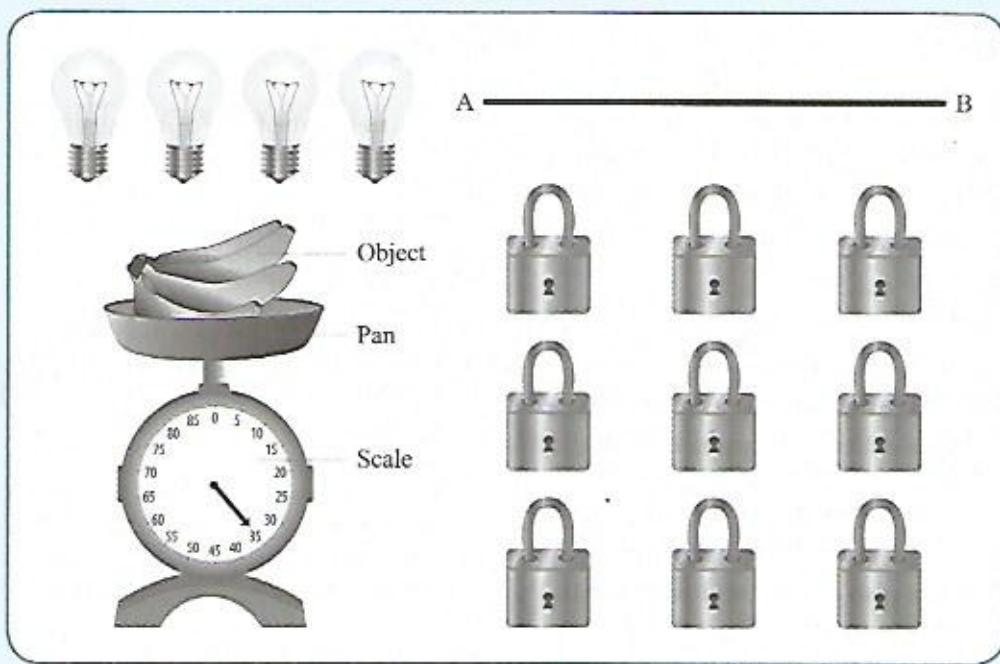
Activities are tasks where you apply the knowledge and skills you learnt in a section.



Sub-topic	Specific Outcomes
Relative and absolute error	<ul style="list-style-type: none"> Work with relative and absolute errors

Starter activity

Look at the diagram below, then answer the questions that follow.



- 1 Say whether or not each of the following can be determined accurately.
 - the number of light bulbs
 - the length of line AB
 - the mass of the bananas
 - the number of locks
- 2 From your answers to Question 1, write down a statement about:
 - numbers of things (found by counting them)
 - measurements of things (found by measuring them).

Introduction

When we count, then the number is true and accurate. However, if we measure a line as 5 cm, it is not necessarily accurate. This is because all measurements are rounded off to a given number of significant digits or decimal places and are therefore subject to error.

A dimension of 2 cm, to the nearest centimetre, lies between 1.5 cm and 2.5 cm (see Figure 1.1). Similarly, a dimension of 3 cm, to the nearest centimetre, lies between 2.5 cm and 3.5 cm (see Figure 1.2).

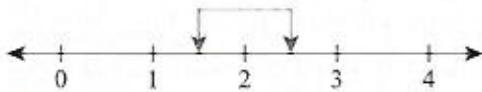


Figure 1.1

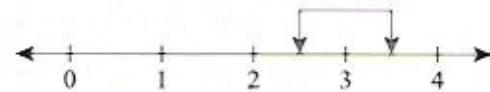


Figure 1.2

We say that there is an error of 0.5 cm in the measurement of the dimensions of the rectangle. If the breadth of the rectangle is b , then b lies between 1.5 cm and 2.5 cm. If the length of the rectangle is l , then l lies between 2.5 cm and 3.5 cm.

We write this as follows:

$$1.5 \text{ cm} \leq b < 2.5 \text{ cm}, \text{ and}$$

$$2.5 \text{ cm} \leq l < 3.5 \text{ cm}$$

These can also be written as follows:

$$b = 2 \pm 0.5 \text{ cm}, \text{ and}$$

$$l = 3 \pm 0.5 \text{ cm}$$

In each case, the smaller number is called the **lower bound** and the bigger number is called the **upper bound**. So, if $1.5 \text{ cm} \leq b < 2.5 \text{ cm}$, 1.5 cm is the lower bound and 2.5 cm is the upper bound. The lower bound and the upper bound are also called the **lower limit** and the **upper limit** respectively.

In general, if a dimension, d , is given to the nearest n , then the error is $\frac{n}{2}$.

New words

dimension: a measurement

lower bound: the smallest number of the interval within which a dimension can fall

upper bound: the biggest number of the interval within which a dimension can fall

lower limit: another name for a lower bound

upper limit: another name for an upper bound

Worked example 1

1 Find the error in each of the following dimensions.

- 340 m, to the nearest 10 m
- 3 600 g, to the nearest 100 g
- 12 ℓ, to the nearest litre
- 10.4 seconds, to the nearest tenth of a second

Worked example 1 (continued)

2 A dimension is stated as $4.3 \text{ cm} \pm 0.05 \text{ cm}$. Find:

- the lower bound
- the upper bound of the dimension.

3 The length and breadth of a rectangle, given to the nearest centimetre, are 15 cm and 10 cm respectively. Find:

- the shortest possible length and the shortest possible breadth of the rectangle
- the longest possible length and the longest possible breadth of the rectangle
- the limits between which the area must lie.

Answers

1 a) $n = 10 \text{ m} \therefore \text{error} = \frac{10 \text{ m}}{2} = 5 \text{ m}$ b) $n = 100 \text{ g} \therefore \text{error} = \frac{100 \text{ g}}{2} = 50 \text{ g}$
 c) $n = 1 \ell \therefore \text{error} = \frac{1 \ell}{2} = 0.5 \ell$ d) $n = 0.1 \text{ s} \therefore \text{error} = \frac{0.1 \text{ s}}{2} = 0.05 \text{ s}$

2 a) The lower bound = $4.3 - 0.05 = 4.25 \text{ cm}$.
 b) The upper bound = $4.3 + 0.05 = 4.35 \text{ cm}$.

3 15 cm, given to the nearest centimetre, lies between 14.5 cm and 15.5 cm.
 10 cm lies between 9.5 cm and 10.5 cm.

- The shortest possible length = 14.5 and the shortest possible breadth = 9.5 cm.
- The longest possible length = 15.5 cm and the longest possible breadth = 10.5 cm.
- The smallest possible area = the shortest possible length \times the shortest possible breadth
 $= 14.5 \times 9.5$
 $= 137.75 \text{ cm}^2$

The largest possible area = the longest possible length \times the longest possible breadth
 $= 15.5 \times 10.5$
 $= 162.75 \text{ cm}^2$

So, the area lies between 137.75 cm^2 and 162.75 cm^2
 (or $137.75 \text{ cm}^2 \leq A < 162.75 \text{ cm}^2$).

Activity 1

1 Which of the following can be found accurately?

- the number of words on this page
- the dimensions of this book
- the mass of a cow
- the number of houses in a village
- the number of teachers in a school

Activity 1 (continued)

- 2 Write down the lower and upper bound of each of the following.
 - a) 5 cm, to the nearest centimetre
 - b) 250 kg, to the nearest 10 kg
 - c) 33.5 ℥, to the nearest tenth of a litre
 - d) 5.81, to the nearest hundredth
- 3 Find the limits between which the following measures must lie. Write your answers in the form $\dots \leq m < \dots$, where m is the given measure.
 - a) 8 cm, to the nearest centimetre
 - b) 0.5 cm, to the nearest millimetre
 - c) 3.5 kg, to the nearest tenth of a kilogram
 - d) 14 m, to the nearest metre
 - e) 1 000 km, to the nearest 10 km
 - f) 52 min, to the nearest 10 seconds
 - g) 20.15 g, to the nearest hundredth of a gram
 - h) 55 m, to the nearest 5 m
 - i) 1 450 km, to the nearest 50 km
 - j) 0.05 kg, to the nearest gram
- 4 If the length of a rod is given as $12 \text{ cm} \pm 0.5 \text{ cm}$, find:
 - a) the shortest possible length of the rod
 - b) the longest possible length of the rod.

Did you know?

We use the SI system of measurement, which is based on seven units of measurement. These are the metre, kilogram, second, Kelvin, ampere, mole and the candela. All the other units of measurement are derived from these seven basic units. For example, the unit of force is the newton, which is expressed in metre kilogram/second², or m.kg.s⁻².

Absolute, relative and percentage errors

The **absolute error** of a dimension is the absolute difference between the true value and the recorded value.

$$\text{Absolute error} = |\text{recorded value} - \text{true value}|$$

The **relative error** of a dimension is the ratio of the absolute error to the true value.

$$\text{Relative error} = \frac{\text{absolute error}}{\text{true value}}$$

The **percentage error** states the relative error as a percentage.

$$\begin{aligned}\text{Percentage error} \\ = \text{relative error} \times 100\%\end{aligned}$$

OR Percentage error

$$= \frac{\text{absolute error}}{\text{true value}} \times 100\%$$

Note

The notation $|x|$ means "the absolute value of x ". The absolute value of a number is the number without its sign. So, $|+6| = 6$ and $|-6| = 6$.

New words

absolute error: the absolute difference between the true value and the recorded value of a dimension

relative error: the ratio of the absolute error to the true value of a dimension

percentage error: the relative error, written as a percentage

Did you know?

There are many careers in which an understanding of errors is extremely important.

For example, a civil engineer who designs a bridge must be very aware of possible errors in measurements, because the bridge must be safe for people to use and strong enough to carry heavy loads of traffic.

In engineering, tolerance is the limits of acceptable error in a measurement for a structure to serve its function properly.



The Victoria Falls bridge on the border of Zambia and Zimbabwe

Worked example 2

- The true value of the length of a rectangle is 10 cm. If this is recorded as 10.2 cm, find:
 - the absolute error
 - the relative error.
- A mass of 24 kg is recorded as 24.3 kg. Find:
 - the relative error
 - the percentage error.
- The length of one side of a square is measured correct to the nearest millimetre. If the upper bound area of the square is 5.0625 cm^2 , calculate:
 - the upper bound length of one side of the square
 - the lower bound length of one side of the square.

Answers

$$\begin{aligned} 1 \text{ a) absolute error} &= |\text{recorded value} - \text{true value}| \\ &= |10.2 - 10| \text{ cm} \\ &= 0.2 \text{ cm} \end{aligned}$$

$$\begin{aligned} \text{b) relative error} &= \frac{\text{absolute error}}{\text{true value}} \\ &= \frac{0.2 \text{ cm}}{10 \text{ cm}} \\ &= 0.02 \end{aligned}$$

$$\begin{aligned} 2 \text{ a) absolute error} &= |24.3 - 24| \text{ kg} \\ &= 0.3 \text{ kg} \end{aligned}$$

$$\begin{aligned} \therefore \text{relative error} &= \frac{\text{absolute error}}{\text{true value}} \\ &= \frac{0.3 \text{ kg}}{24 \text{ kg}} \\ &= 0.0125 \end{aligned}$$

$$\begin{aligned} \text{b) percentage error} &= \text{relative error} \times 100\% \\ &= 0.0125 \times 100\% \\ &= 1.25\% \end{aligned}$$

Worked example 2 (continued)

3 a) $l^2 = A$, where l = length and A = area

$$\begin{aligned}\therefore l &= \sqrt{A} \\ &= \sqrt{5.0625} \\ &= 2.25\end{aligned}$$

The upper bound length is 2.25 cm.

b) The length is given correct to 1 mm, which is 0.1 cm.

$$\therefore \text{error} = \frac{0.1}{2} = 0.05$$

$$\therefore \text{the measured value} = 2.25 - 0.05 = 2.2$$

$$\therefore \text{the lower bound length} = 2.2 - 0.05 = 2.15 \text{ cm}$$

Activity 2

Work in groups of four or five for this activity.

- 1 Each one of you should take a ruler and measure the length of the front cover of this textbook (from top to bottom), correct to the nearest millimetre.
 - a) Write down your measurements.
 - b) Compare your measurements with those of the other members of your group, and calculate the average length that your group measured.
 - c) If the actual length of the front cover of this textbook is 240 mm, calculate:
 - i) the absolute error,
 - ii) the relative error and
 - iii) the percentage error of your own measurement
 - iv) the absolute error,
 - v) the relative error and
 - vi) the percentage error of your group's average measurement.
- 2 Next, each one of you should take a ruler and measure the length of your desk, correct to the nearest millimetre.
 - a) Write down your measurements.
 - b) Compare your measurements with those of the other members of your group. Did these measurements differ more widely than your group's measurements in Question 1? If so, try to think of practical reasons for this.

Activity 3

- 1 The true value of the length of a rectangle is 4 cm. If this is recorded as 4.4 cm, find:
 - a) the absolute error
 - b) the relative error
 - c) the percentage error.
- 2 If a square has a side of length 84 cm, correct to the nearest centimetre, find:
 - a) the shortest possible length of the side
 - b) the longest possible length of the side
 - c) the smallest possible area of the square
 - d) the largest possible area of the square.

Activity 3 (continued)

3 If an athlete takes 12.4 seconds, correct to the nearest tenth of a second, to complete a 100 m race, find:

- the lower bound time
- the upper bound time
- the lower bound average speed of the athlete in m/s, correct to three decimal places.
- the upper bound average speed of the athlete in m/s, correct to three decimal places.

4 The length of a side of a square is measured correct to the nearest tenth of a millimetre. If the upper bound area of the square is 692.7424 cm^2 , calculate:

- the measured length
- the upper bound length
- the lower bound length.

5 The length and breadth of a rectangle are given as 20 cm and 10 cm respectively, correct to the nearest centimetre. Find:

- the shortest possible perimeter
- the longest possible perimeter
- the smallest possible area of the rectangle
- the largest possible area of the rectangle.

6 The costs of seven different phases of a construction project were as follows: K4 564 238.58, K7 073 788.82, K763 687.19, K5 972 243.32, K1 042 219.69, K7 976 142.57 and K9 454 704.61. Estimate the total cost of the project by first rounding each of the seven costs off to the nearest:

- K1 000 000
- K100 000
- K10 000

7 If a circle of radius 16 cm has the radius recorded as 16.1 cm, correct to the nearest millimetre, find:

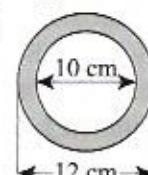
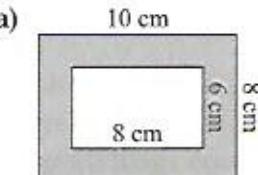
- the absolute error
- the relative error
- the percentage error.

8 If the radius of a circle is given as 12.5 cm, correct to the nearest millimetre, find:

- the limits in the form $\square \leq r < \square$ between which the radius, r , must lie
- the limits in the form $\square \leq C < \square$ between which the circumference, C , must lie, correct to the nearest millimetre
- the limits in the form $\square \leq A < \square$ between which the area, A , must lie, correct to the nearest square millimetre.

9 In the diagrams below, the area of the shaded region is $A \text{ cm}^2$. All dimensions are given to the nearest centimetre. In each case, calculate, correct to the nearest square centimetre:

- the lower limit of A
- the upper limit of A .



Summary

Relative and absolute error

- All dimensions (measurements) are rounded off and are therefore subject to error. For example, a length of 15 cm, correct to the nearest centimetre, may have an error of 0.5 cm. So the actual length will lie between 14.5 cm and 15.5 cm.
- In this case, the lower bound (lower limit) is 14.5 cm and the upper bound (upper limit) is 15.5 cm.
- We write this as follows: $14.5 \text{ cm} \leq l < 15.5 \text{ cm}$, or $l = 15 \pm 0.5 \text{ cm}$.
- In general, if a dimension, d , is given to the nearest n , then the error is $\frac{n}{2}$.
- The notation $|x|$ means "the absolute value of x ". The absolute value of a number is the number without its sign. So, $|+6| = 6$ and $|-6| = 6$.
- The **absolute error** of a dimension is the absolute difference between the true value and the recorded value. So, absolute error = $|\text{recorded value} - \text{true value}|$.
- The **relative error** of a dimension is the ratio of the absolute error to the true value. So, relative error = $\frac{\text{absolute error}}{\text{true value}}$.
- The **percentage error** states the relative error as a percentage. So, percentage error = relative error $\times 100\%$ OR percentage error = $\frac{\text{absolute error}}{\text{true value}} \times 100\%$.

Revision exercises (remedial)

- 1 Say whether or not each of the following can be found accurately. Give a reason each time.
 - a) the number of learners in your class
 - b) the distance between Lusaka and Johannesburg
 - c) the mass of a brick
 - d) the number of grains of rice in a pot

Revision exercises

- 2 Find the lower and upper bound of each of the following measures.
 - a) 10 kg, to the nearest kilogram
 - b) 500 km, to the nearest kilometre
 - c) 100 m, to the nearest 10 metres
 - d) 42.2 cm, to the nearest tenth of a centimetre

(1)

(1)

(1)

(1)

(1)

(1)

(1)

e) 145 kg, to the nearest 5 kilograms (2)
f) 1 500 km, to the nearest 100 kilometres (2)
g) 4.24 cm, to the nearest hundredth of a centimetre (2)

3 The true value of the mass of an object is 51 g. If this is recorded as 50 g, find:
a) the absolute error (2)
b) the relative error. (2)

4 A capacity of 3.25ℓ is recorded as 3.3ℓ . Find:
a) the relative error (2)
b) the percentage error. (2)

5 The length and breadth of a rectangle are given as 10 cm and 8 cm respectively, both to the nearest centimetre. Calculate:
a) the lower and upper bound length (2)
b) the lower and upper bound breadth (2)
c) the lower and upper bound perimeter (2)
d) the lower and upper bound area. (2)

6 The length of one side of a square is measured correct to two decimal places. The upper limit area of the square is given as $2\ 077.5364 \text{ cm}^2$. Find:
a) the upper limit length of one side of the square (2)
b) the lower limit length of one side of the square. (2)

7 The masses of objects A and B, given to the nearest kilogram, are 14 kg and 25 kg. Find:
a) the lower and upper bound mass of object A (2)
b) the lower and upper bound mass of object B (2)
c) the largest combined mass of objects A and B (2)
d) the smallest combined mass of objects A and B (2)
e) the smallest difference between the two masses (2)
f) the largest difference between the two masses. (2)

Total marks: 50

Assessment exercises

(2)

(2)

(2)

(2)

(2)

(2)

1 Write down the lower and upper limit of each of the following.
a) 20 cm, to the nearest centimetre (2)
b) 4.5 kg, to the nearest tenth of a kilogram (2)
c) $4\ 580 \ell$, to the nearest 10 litres (2)

2 The true value of the length of a field is 395 m. If this is recorded as 400 m, find:
a) the absolute error (2)
b) the relative error (2)
c) the percentage error. (2)

Revision and assessment continued

3 A train completes a 700 km journey in 20 hours. Both measures are given to the nearest 10 units.

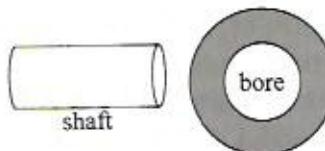
Calculate:

- a) the lower and upper bound distance (2)
- b) the lower and upper bound time (2)
- c) the lower and upper bound speed. (4)

4 If the radius of a circle is given as 25.2 cm, correct to the nearest millimetre, find:

- a) the limits of the radius, r , in the form $\dots \leq r < \dots$ (2)
- b) the limits of the circumference, C , in the form $\dots \leq C < \dots$ (2)
- c) the limits of the area, A , in the form $\dots \leq A < \dots$ (2)

5 The diagram below shows a bore and a shaft. The shaft is designed to fit in the bore. The shaft has a diameter of 12.5 cm and the bore has a diameter of 12.7 cm, both correct to one decimal place. Both the bore and the shaft are 120 cm long, correct to one decimal place.



Find:

- a) the lower and upper bound diameter of the shaft (2)
- b) the lower and upper bound diameter of the bore (2)
- c) the smallest volume of the space between the bore and the shaft when they have been assembled (4)
- d) the largest volume of the space between the bore and the shaft when they have been assembled. (4)

6 In the construction of a highway, the depth of the drainage layer thrown must be 72 cm, with a tolerance of 5 cm. At various points along the highway, the depth of the drainage layer is tested. Which of the following measurements are unacceptable in this case?

67 cm, 79 cm, 71 cm, 62 cm, 81 cm, 70 cm, 95 cm, 76 cm (3)

7 You already know that if a measurement, m , of 3 kg has been given correct to the nearest kilogram, then we write this as $2.5 \text{ kg} \leq m < 3.5 \text{ kg}$. Why do we not write this as $2.5 \text{ kg} \leq m \leq 3.5 \text{ kg}$?

(Hint: Think about the rules for rounding off numbers.) (4)

Total marks: 45



	Sub-topics	Specific Outcomes
(2) (2) (4)	Arithmetic progressions	<ul style="list-style-type: none"> Identify an arithmetic progression (AP). Find the nth term of an arithmetic progression. Find the arithmetic mean of an arithmetic progression. Find the sum of an arithmetic progression.
(2) (2) (2)	Geometric progressions	<ul style="list-style-type: none"> Identify a geometric progression (GP). Find the nth term of a geometric progression. Find the geometric mean of a geometric progression. Find the sum of a geometric progression. Find the sum to infinity of a geometric progression.

Starter activity

1 Write down the next three terms in each of the following sequences. In each case, state the rule for finding the next term of the sequence.

a) 200, 100, 50, 25, ...
 b) 3, 4.5, 6, 7.5, ...
 c) 2, -5, -12, -19, ...
 d) 3, -6, 12, -24, ...

2 Belita's mother has some cows and chickens. Belita takes turns with her two brothers to deliver fresh milk and eggs to the market every morning. Her two deliveries in September were on Wednesday the 2nd and Saturday the 5th. Use the calendar to list the other dates on which it will be Belita's turn to make the deliveries for the rest of the month.



September						
S	M	T	W	T	F	S
			1	2	3	4
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	26
27	28	29	30			

Introduction

A sequence is an ordered set of numbers, where a pattern exists. A **term** is a number in a sequence.

So, in the sequence, 3, 6, 9, 12, 15, ..., the first term is 3, the second term is 6, the fifth term is 15, and so on. The terms are separated by commas. The three dots at the end of each sequence show that the sequence carries on infinitely.

New words

sequence: an ordered set of numbers, where a pattern exists

term: a number in a sequence

Identify an arithmetic progression (AP)

Consider the following sequences:

1, 4, 7, 10, The first term is 1 and the remaining terms are formed by adding 3 each time.

17, 11, 5, -1, The first term is 17 and the remaining terms are formed by adding -6 (or subtracting 6) each time.

The sequences above are examples of arithmetic progressions. An **arithmetic progression** is a sequence in which each term is formed by adding a constant amount (which may be positive or negative) to the previous term. This constant amount is called a **common difference**. We use the letter a for the first term and the letter d for the common difference. The abbreviation of the term "arithmetic progression" is AP. Another name for an arithmetic progression is an **arithmetic sequence**.

New words

arithmetic progression: a sequence in which each term is formed by adding a constant amount to the previous term

common difference: the constant difference between any two consecutive terms in an arithmetic progression

AP: the abbreviation of the term "arithmetic progression"

arithmetic sequence: another name for an arithmetic progression

Activity 1

- For each of the following sequences, say whether or not it is an arithmetic progression.
 - 5, 6, 7, 8, ...
 - 3, 6, 12, 24, ...
 - 1, 2, 4, 7, ...
 - 58, -61, -64, -67, ...
 - 0.4, 0.9, 1.4, 1.9, ...
 - $\frac{1}{2}, \frac{3}{4}, 1, 1\frac{1}{4}, \dots$
- For each of the APs that you identified in Question 1, write down the value of a and d .

Find the n th term of an arithmetic progression

First term	$T_1 = a$	$= a + 0d$	$= a + (1 - 1)d$
Second term	$T_2 = a + d$	$= a + 1d$	$= a + (2 - 1)d$
Third term	$T_3 = a + d + d$	$= a + 2d$	$= a + (3 - 1)d$
Fourth term	$T_4 = a + d + d + d$	$= a + 3d$	$= a + (4 - 1)d$
Fifth term	$T_5 = a + d + d + d + d$	$= a + 4d$	$= a + (5 - 1)d$
...			
n th term	$T_n = a + d + d + \dots + d$	$= a + (n - 1)d$	
	$\underbrace{d + d + \dots + d}_{(n - 1) \text{ times}}$		

Notice the pattern in the way the terms are formed.

The formula for the n th term of an AP with first term a and common difference d is:

$$T_n = a + (n - 1)d$$

Note

In an AP, the difference between any two consecutive terms is d . This gives us the following useful formula:
 $d = T_2 - T_1 = T_3 - T_2 = \dots = T_n - T_{n-1}$

Worked example 1

- Find the next three terms in each of the following sequences.
 - $9, 12, 15, 18, \dots$
 - $r + 16, r + 9, r + 2, \dots$
- Find the 37th term in the sequence $16, 7, -2, \dots$
- Find a formula for the n th term of the AP: $24, 35, 46, \dots$
- Find the first four terms of the AP of which $T_{12} = 89$ and $T_{19} = 131$.

Answers

- $d = 12 - 9 = 3$ (or $15 - 12 = 3$ or $18 - 15 = 3$)
 $T_5 = 18 + 3 = 21; T_6 = 21 + 3 = 24; T_7 = 24 + 3 = 27$
 The next three terms are 21, 24 and 27.
 - $d = (r + 9) - (r + 2) = 7$.
 $T_4 = r + 2 + (-7) = r - 5; T_5 = r - 5 + (-7) = r - 12; T_6 = r - 12 + (-7) = r - 19$
 The next three terms are $r - 5, r - 12$ and $r - 19$.
- 2 $a = 16$ and $d = 7 - 16 = -9$
- $T_n = a + (n - 1)d$
 $\therefore T_{37} = 16 + (37 - 1)(-9) = -308$
- $T_{12} = a + (12 - 1)d = 89$
 $\therefore a + 11d = 89$
- $T_{19} = a + (19 - 1)d = 131$
 $\therefore a + 18d = 131$
- Subtract ① from ②: $18d - 11d = 131 - 89$
 $\therefore 7d = 42$
 $\therefore d = 6$
- Substitute ③ into ①: $a + 11(6) = 89$
 $\therefore a = 89 - 66 = 23$

The first four terms are 23, 29, 35 and 41.

Activity 2

- Write down the next three terms in each of the following APs.
 - 23, 29, 35, 41, ...
 - 10, -7, -4, ...
 - $1\frac{1}{2}$, 3, $4\frac{1}{2}$, ...
- Find the value of a , d and T_{31} of each of the following APs.
 - 16, 10, 4, ...
 - 1, 5, 9, ...
 - $\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \dots$
 - $\frac{1}{2}, 0, -\frac{1}{2}, \dots$
- If the first term of an AP is 9 and the 20th term is -29, find the common difference.
- Find the 58th term of the AP: -10, -7, -4, ...
- If the second term of an AP is -8 and the common difference is -6, find:
 - the first term
 - a formula for the n th term.
- Find the general term, T_n , for each of the following APs.
 - 10, 8, 6, ...
 - 0, 11, 22, ...
 - 1, 4, 9, ...
 - $\frac{2}{3}, \frac{1}{3}, 0, \dots$
- Show, by calculation, whether or not -268 is a term in the AP 86, 69, 52, ...
- In the AP 12, 23, 34, ..., which term is 1 200?
- A mining firm sold K200 000 worth of copper in the first year of its operation. The sales increased by K24 000 each year. Calculate the value of their sales in the seventh year.
- A car was purchased for K18 800 and it is estimated that it will lose value at the rate of K185 per month. How many months will pass before it is valued at K9 920?



A miner working in a copper mine

Find the arithmetic mean of an arithmetic progression

If a , b and c are three consecutive terms in an AP, then b is the **arithmetic mean** of a and c .

The formula for the arithmetic mean, b , is as follows: $b = \frac{1}{2}(a + c)$

New word

arithmetic mean: the middle term of three consecutive terms in an AP

Worked example 2

- Find x and y if 16, x , 6, y form an arithmetic progression.
- Form an arithmetic progression that has five arithmetic means between -7 and 23.

Answers

- x is the arithmetic mean of 16 and 6.

$$\therefore x = \frac{1}{2}(16 + 6) = 11$$

The first three terms are 16, 11, 6. This gives a common difference of -5.

$$\therefore y = 6 + (-5) = 1$$

Worked example 2 (continued)

2 This AP is of the form $-7, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}, 23$.

$$a = -7 \text{ and } T_1 = 23$$

$$T_c = a + (n-1)d$$

$$\therefore -7 + (7 - 1)d = 23$$

$$\therefore 6d = 30$$

$d = 5$

$$T_1 = -7 + 5 = -2; T_2 = -2 + 5 = 3; T_3 = 3 + 5 = 8; T_4 = 8 + 5 = 13; T_5 = 13 + 5 = 18$$

The AP is $-7, -2, 3, 8, 13, 18, 23$

Activity 3

- 1 Find the arithmetic mean of the following pairs of numbers:
 - a) 64 and 67
 - b) -85 and -95
- 2 Form an arithmetic progression that has two arithmetic means between 3 and -21.
- 3 Form an arithmetic progression that has four arithmetic means between 7.5 and 35.

Find the sum of an arithmetic progression

An **arithmetic series** is the sum of an arithmetic progression. So, an arithmetic series is an AP with plus signs written between the terms.

New word

arithmetic series: the sum of an arithmetic progression

The table on the right shows are examples of arithmetic progressions and their corresponding arithmetic series.

Arithmetic progression	Arithmetic series
$1, 4, 7, 10, 13, \dots$	$1 + 4 + 7 + 10 + 13 + \dots$
$\frac{1}{2}, \frac{1}{4}, 0, -\frac{1}{4}, -\frac{1}{2}, \dots$	$\frac{1}{2} + \frac{1}{4} + 0 + \left(-\frac{1}{4}\right) + \left(-\frac{1}{2}\right) + \dots$

To develop a formula for the sum of an AP, write an expression for S_n .

$$S_n = a + (a + d) + (a + 2d) + \dots + [a + (n-2)d] + [a + (n-1)d] \quad \text{①}$$

Reverse the terms in this formula:

$$S_n = [a + (n-1)d] + [a + (n-2)d] + \dots + (a + 2d) + (a + d) + a \quad (2)$$

Add ① and ②: $2S_n = [2a + (n-1)d] + [2a + (n-1)d] + \dots + [2a + (n-1)d]$

n times

$$\therefore 2S_n = n[2a + (n-1)d]$$

$$\therefore S_n = \frac{n}{2} [2a + (n-1)d]$$

So, the formula for the sum of the first n terms in an AP, where a is the first term and d is the common difference, is: $S_n = \frac{n}{2} [2a + (n-1)d]$.

We can also derive another formula for the sum of the first n terms in an AP, in terms of n , a and l , where l is the last term in the AP.

The last term of any AP with n terms is the n th term or T_n , and so $l = a + (n-1)d$.

$$\begin{aligned}\therefore S_n &= \frac{n}{2} [2a + (n-1)d] \\ &= \frac{n}{2} [a + a + (n-1)d] \\ &= \frac{n}{2} [a + l]\end{aligned}$$

So, the formula for the sum of the first n terms in an AP, where a is the first term and l is the last term, is: $S_n = \frac{n}{2} [a + l]$

Worked example 3

- 1 Calculate the sum of the integers from 10 to 45.
- 2 The first three terms of an AP are 5, 7 and 9 and the sum of its terms is 480. Show that the AP has 20 terms.

Answers

1 The number of terms is

$$45 - 10 + 1 = 36.$$

$$S_n = \frac{n}{2} (a + l)$$

$$\therefore S_{36} = \frac{36}{2} (10 + 45) = 990$$

2 $S_n = \frac{n}{2} [2a + (n-1)d]$

$$\therefore S_{20} = \frac{20}{2} [2(5) + 19(2)] = 480$$

So, the AP has 20 terms.

Activity 4

- 1 Find the sum of the first:
 - 15 terms of the AP: 11, 5, -1, -7, ...
 - 32 terms of the AP: 0.5, 0.75, 1, ...
 - 13 terms of the AP: $2, 2\frac{1}{4}, 2\frac{1}{2}, \dots$
 - 20 terms of the AP: -81, -78, -75, ...
 - 25 terms of the AP: 17, 15, 13, ...
 - n terms of the AP: 7, 11, 15, ...
- 2 The first term of an AP is 126 and the 30th term is 387. Find S_{30} .
- 3 S_8 of an AP is 116 and S_9 is 117.
 - Write down the value of T_9 .
 - Calculate the values of a and d .
- 4 Calculate the sum of all the odd numbers from 19 to 399.
- 5 Naomi is planning her revision programme for her term exams. On day 1, she plans to study for one hour. Each day thereafter, she plans to increase her study time by ten minutes.
 - For how long will she study on the 14th day?
 - How long will she have studied altogether by then?

SUB-TOPIC 2 Geometric progressions

Identify a geometric progression (GP)

Consider the following sequences:

1, 2, 4, 8, 16, The first term is 1 and the remaining terms are formed by multiplying by 2 each time.

1 000, 100, 10, 1, 0.1, The first term is 1 000 and the remaining terms are formed by multiplying by 0.1 (or dividing by 10) each time.

The sequences above are examples of geometric progressions. A **geometric progression** is a sequence in which each term is formed by multiplying the previous term by a constant amount.

This constant amount is called a **common ratio**. We use the letter a for the first term and the letter r for the common ratio.

The abbreviation of the term "geometric progression" is GP. Another name for a geometric progression is a **geometric sequence**.

New words

geometric progression: a sequence in which each term is formed by multiplying the previous term by a constant amount

common ratio: the constant ratio between any two consecutive terms in a geometric progression

GP: the abbreviation of the term "geometric progression"

geometric sequence: another name for a geometric progression

Did you know?

Arithmetic and geometric progressions are used in many careers in real life. For example, they are used in banking to calculate interest on investments and loans, as well as in the fields of manufacturing and engineering.

Activity 5

- For each of the following sequences, say whether or not it is a geometric progression.
 - 10, 11, 12, 13, ...
 - 3, 6, 12, 24, ...
 - 1, 2, 4, 7, ...
 - 100, 50, -25, 12.5, ...
 - 0.4, 0.44, 0.444, 0.4444, ...
 - $\frac{9}{100}, \frac{3}{10}, 1, \frac{10}{3}, \dots$
- For each of the GPs that you identified in Question 1, write down the values of a and r .

Find the n th term of a geometric progression

First term	$T_1 = a$	$= a$	$= ar^{1-1}$
Second term	$T_2 = a \times r$	$= ar$	$= ar^{2-1}$
Third term	$T_3 = a \times r \times r$	$= ar^2$	$= ar^{3-1}$
Fourth term	$T_4 = a \times r^2 \times r$	$= ar^3$	$= ar^{4-1}$
Fifth term	$T_5 = a \times r^3 \times r$	$= ar^4$	$= ar^{5-1}$
...			
n th term	$T_n = a \times r \times r \times \dots \times r$ <small>($n-1$) times</small>	$= ar^{n-1}$	

Notice the pattern in the way the terms are formed.

The formula for the n th term of a GP with first term a and common ratio r is: $T_n = ar^{n-1}$.

Note

In a GP, the ratio between any two consecutive terms is r . This gives us the following useful formula:

$$r = \frac{T_2}{T_1} = \frac{T_3}{T_2} = \dots = \frac{T_n}{T_{n-1}}$$

Worked example 4

- For each of the following GPs, find a formula for the n th term.
 - 4, 12, 36, 108, ...
 - 256, 128, 64, 32, ...
 - $\frac{5}{6}, 1, \frac{6}{5}, \dots$
- If $x - 2$, x and $x + 3$ are the first three terms of a GP, find:
 - the value of x
 - the common ratio.
- The fifth and eighth terms of a GP are 80 and 640 respectively.
 - Find the values of r and a .
 - Write down the first five terms of the GP.

Answers

1 a) $r = 12 \div 4 = 3$
 $\therefore T_n = ar^{n-1} = 4(3)^{n-1}$

b) $r = \frac{128}{256} = \frac{1}{2}$
 $\therefore T_n = ar^{n-1} = 256\left(\frac{1}{2}\right)^{n-1} = 2^{8-(n-1)} = 2^{9-n}$

c) $r = 1 \div \frac{5}{6} = \frac{6}{5}$
 $\therefore T_n = ar^{n-1} = \frac{5}{6}\left(\frac{6}{5}\right)^{n-1} = \left(\frac{6}{5}\right)^1\left(\frac{6}{5}\right)^{n-1} = \left(\frac{6}{5}\right)^{n-2}$

2 a) $r = \frac{x}{(x-2)} = \frac{(x+3)}{x}$
 $\therefore x^2 = x^2 + x - 6$
 $\therefore x = 6$

b) The three terms are 4, 6, and 9, $\therefore r = \frac{6}{4} = \frac{3}{2}$.

3 a) $T_n = ar^{n-1}$ $T_8 = ar^{8-1}$
 $\therefore T_5 = ar^{5-1}$ $\therefore ar^7 = 640 \quad \textcircled{2}$
 $\therefore ar^4 = 80 \quad \textcircled{1}$
 Divide $\textcircled{2}$ by $\textcircled{1}$: $\therefore \frac{ar^7}{ar^4} = \frac{640}{80}$
 $\therefore r^3 = 8$
 $\therefore r = 2 \quad \textcircled{3}$

Substitute $\textcircled{3}$ into $\textcircled{1}$: $\therefore a(2)^4 = 80$
 $\therefore 16a = 80$
 $\therefore a = 5$

b) The first five terms are 5, 10, 20, 40 and 80.

Activity 6

- Write down the first four terms of the GP of which $a = 3$ and $r = 2$.
- Calculate the value of T_{12} of a GP if the first term is 7 and the 6th term is 224.
- Find the formula for the n th term of each of the following GPs.
 - 8, 24, 72, ...
 - $\frac{8}{27}, \frac{4}{9}, \frac{2}{3}, \dots$
 - 64, 16, 4, ...
- Find the first term and the common ratio of a GP of which $T_2 = 12$ and $T_5 = 768$.
- The three numbers $x - 1$, x and $x + 3$ are consecutive terms in a GP. Find:
 - the value of x
 - the common ratio
 - the third term.
- The third and sixth terms of a GP are $\frac{4}{3}$ and $4\frac{1}{2}$ respectively. Find:
 - the common ratio
 - the fifth term.

Find the geometric mean of a geometric progression

If a , b and c are three consecutive terms in a GP, then b is the **geometric mean** of a and c .

$$\text{The common ratio, } r = \frac{b}{a} = \frac{c}{b}$$

$$\therefore b^2 = ac$$

$$\therefore b = \pm \sqrt{ac}$$

New word

geometric mean: the middle term of three consecutive terms in a GP

Worked example 5

- Find two possible values for the geometric mean of 5 and 20.
- Form a GP that has two geometric means between 243 and -9.

Answers

- Let x be the geometric mean of 5 and 20.

$$\therefore \frac{x}{5} = \frac{20}{x}$$

$$\therefore x^2 = 100$$

$$\therefore x = \pm \sqrt{100}$$

$$\therefore x = \pm 10$$
- Let x and y be the geometric means. So, this GP is of the form 243, x , y , -9.

$$\therefore 243 \times r^3 = -9$$

$$\therefore r^3 = \frac{-9}{243} = -\frac{1}{27}$$

$$\therefore r = \sqrt[3]{-\frac{1}{27}} = -\frac{1}{3}$$

$$\therefore x = 243 \times -\frac{1}{3} = -81$$

$$\text{and } y = -81 \times -\frac{1}{3} = 27$$

So, the GP is 243, -81, 27, -9.

Activity 7

- Find two possible values for the geometric mean of:
 - 16 and 36
 - $-\frac{1}{4}$ and -4
- Form a geometric progression that has two geometric means between
 - 1 029 and -3
 - $\frac{2}{k}$ and $\frac{1}{4k^4}$
- Insert three geometric means between 5 and 80. Find all the possibilities.

Find the sum of a geometric progression

A geometric series is the sum of a geometric progression. So, a geometric series is a GP with plus signs written between the terms.

New words

geometric series: the sum of a geometric progression

To develop a formula for the sum of a GP, write an expression for S_n .

$$S_n = a + ar + ar^2 + ar^3 + ar^4 + \dots + ar^{n-1} \quad \textcircled{1}$$

Then multiply this expression by r :

$$rS_n = ar + ar^2 + ar^3 + ar^4 + \dots + ar^{n-1} + ar^n \quad \textcircled{2}$$

Subtract $\textcircled{2}$ from $\textcircled{1}$:

$$S_n = a + ar + ar^2 + ar^3 + ar^4 + \dots + ar^{n-1}$$

$$-rS_n = ar + ar^2 + ar^3 + ar^4 + \dots + ar^{n-1} + ar^n$$

$$S_n - rS_n = a - ar^n$$

$$\therefore S_n(1 - r) = a(1 - r^n)$$

$$\therefore S_n = \frac{a(1 - r^n)}{(1 - r)}$$

So, the formula for the sum of the first n terms in a GP, where a is the first term and r is the common ratio, is: $S_n = \frac{a(1 - r^n)}{(1 - r)}$

This form is convenient to use when $|r| < 1$.

This formula can also be written in the form:

$$S_n = \frac{a(r^n - 1)}{(r - 1)}$$

This form is convenient to use when $|r| > 1$.

Notice how all the terms cancel out, with the exception of the first and the last terms.

Note

The notation $|r|$ means "the absolute value of r ". The **absolute value** of a number is the number without its sign. So, $|+6| = 6$ and $|-6| = 6$.

New word

absolute value: the numeric value of a number without its sign

Worked example 6

- Find S_{10} of the GP: $\frac{1}{2}, \frac{1}{3}, \frac{2}{9}, \dots$ to three significant figures.
- If S_7 of a GP is 6 558 and the constant ratio is 3, calculate the value of the first term.

Answers

1 $a = \frac{1}{2}$ and $r = \frac{2}{3}$

$$S_n = \frac{a(1 - r^n)}{(1 - r)}$$

(Use this form, because $|r| < 1$.)

$$\therefore S_{10} = \frac{\frac{1}{2}(1 - \frac{2^{10}}{3})}{(1 - \frac{2}{3})}$$

$$= \frac{3}{2}(1 - 0.01734)$$

$$= 1.47$$

2 $S_n = \frac{a(r^n - 1)}{(r - 1)}$

(Use this form, because $|r| > 1$.)

$$\therefore S_7 = \frac{a(3^7 - 1)}{(3 - 1)}$$

$$\therefore \frac{a(2187 - 1)}{2} = 6558$$

$$\therefore a(2186) = 13116$$

$$\therefore a = \frac{13116}{2186} = 6$$

Activity 8

- Find the first three terms of the GP of which the common ratio is $-\frac{2}{3}$ and S_6 is 133.
- Find the sum of the first ten terms of the GP: 3, 12, 48,
- For the GP: 2, 6, 18, ..., find:
 - a formula for S_n
 - the sum of the first fifteen terms.
- The first term of a GP is -20 and the tenth term is 10 240. Find S_{11} of this GP.
- The attendance at a school concert over four evenings forms a GP. If the constant ratio is 1.2 and 1 342 people attended altogether, how many people attended the concert on each of the four evenings?

Find the sum to infinity of a geometric progression

Any progression that has an infinite number of terms is an **infinite progression**. Under certain circumstances, it is possible to calculate the sum to infinity of a geometric progression.

New word

infinite progression: a progression that has an infinite number of terms

Worked example 7

Consider the GP: 3, $\frac{3}{2}$, $\frac{3}{4}$,

- Find the values of a and r .
- Find the formula for S_n for this GP.
- Calculate the value of S_n correct to eight decimal places, for $n = 5, 15$ and 25 .
- What do you notice about the value of S_n as the value of n increases?

Answers

a) $a = 3$ and $r = \frac{1}{2}$

b) $S_n = \frac{a(1 - r^n)}{(1 - r)}$
 $= \frac{3[1 - (\frac{1}{2})^n]}{1 - \frac{1}{2}} = \frac{3[1 - (\frac{1}{2})^n]}{\frac{1}{2}} = 6(1 - \frac{1}{2^n})$

c) $S_5 = 6(1 - \frac{1}{2^5}) = 5.81250000$

d) As the value of n becomes larger and larger, S_n becomes closer and closer to 6.

$S_{15} = 6(1 - \frac{1}{2^{15}}) = 5.99981690$

$S_{25} = 6(1 - \frac{1}{2^{25}}) = 5.99999982$

In the example above, we say that the limiting value of S_n is 6 as n , the number of terms, tends to infinity.

We write this as follows: $\lim_{n \rightarrow \infty} 6(1 - \frac{1}{2^n}) = 6$.

This limiting value of S_n is called the **sum to infinity** (S_∞) of the GP.

Note

∞ is not a number, but a symbol that denotes that n and S_n are increasing continuously without limit.

The formula for the sum of the first n terms of a GP with the first term a and the common ratio r is:

$$S_n = \frac{a(1-r^n)}{(1-r)} = \frac{a}{(1-r)} - \frac{ar^n}{(1-r)}.$$

If $|r| < 1$, that is if $-1 < r < 1$, then r^n will become very small as n becomes large. If r^n becomes very small, then $\frac{ar^n}{(1-r)}$ will become very small, that is $\frac{ar^n}{(1-r)} \rightarrow 0$ as $n \rightarrow \infty$. The arrow \rightarrow is read as "tends to". If this is the case, then $S_n \rightarrow \frac{a}{(1-r)}$ as $n \rightarrow \infty$.

The sum to infinity of a GP with first term a and common ratio r is:

$$S_\infty = \frac{a}{(1-r)} \text{ if and only if } -1 < r < 1.$$

If a GP has $-1 < r < 1$, then we say that the GP converges. If not, then we say that it diverges. A GP has a sum to infinity if and only if it is a **convergent GP**. A divergent GP does not have a sum to infinity.

New words

sum to infinity: the limiting value of the sum of a GP, as the number of terms tends to infinity

convergent GP: a GP of which the terms approach a limit (consecutive terms become closer together)

divergent GP: a GP of which the terms do not approach a limit (consecutive terms become further apart)

Worked example 8

1 Find the sum to infinity of this GP: $1, -\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \dots$

2 The sum to infinity of a GP is 6. If the first term is 2, find the common ratio.

Answers

1 $|r| = \left| -\frac{2}{3} \right| < 1$

$$\therefore S_\infty = \frac{a}{(1-r)} = \frac{1}{(1 - (-\frac{2}{3}))} = \frac{3}{5}$$

2 $S_\infty = \frac{a}{(1-r)} = \frac{2}{(1-r)} = 6$

$$\therefore 6 - 6r = 2$$

$$\therefore r = \frac{2}{3}$$

Activity 9

1 Show, by calculation, whether or not each of the following GPs converges.

a) $8, 4, 2, \dots$ b) $2, -10, 50, \dots$

c) $45, 30, 20, \dots$ d) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \dots$

e) $0.0001, 0.003, 0.09, \dots$ f) $4, 0.4, 0.04, \dots$

2 Calculate the sum to infinity for each convergent GP in Question 1.

3 The sum to infinity of a GP is 64 and its constant ratio is $\frac{3}{4}$. Find the value of:

a) a b) S_6 , correct to three decimal places.

4 If $3x$, $9x^2$ and $27x^3$ are consecutive terms in a GP, find all possible values of x for which the GP converges.

5 A mining company extracted 200 000 tonnes of copper in the first year of its operation. The annual output of the mine decreases by 4% each year. If this trend continues indefinitely, show that the mine will yield a maximum of 5 million tonnes of copper.

Summary

arithmetic

- An arithmetic sequence
- Adding a common difference
- The formula for the n th term: $T_n = a + (n-1)d$
- If a, b and c are in arithmetic progression, then $a + c = 2b$.
- An arithmetic series
- The sum of the first n terms: $S_n = \frac{n}{2}[2a + (n-1)d]$

geometric

- A geometric sequence
- Multiplying by a common ratio
- The formula for the n th term: $T_n = ar^{n-1}$
- If a, b and c are in geometric progression, then $a \cdot c = b^2$.
- A geometric series
- The sum of the first n terms: $S_n = \frac{a(1-r^n)}{1-r}$
- The sum to infinity of a convergent GP: $S_\infty = \frac{a}{1-r}$

Revision

- 1 Say whether each of the following GPs converges.
 - 1, 4, 9, ...
 - 1.7, 2, ...
 - 5, -1, ...
- 2 For each convergent GP in Activity 9, calculate the sum to infinity.
- 3 For each convergent GP in Activity 9, calculate the value of a .

Revision

- 4 For each GP in Question 5, calculate the value of x .
- a) $0.5, 4, 16, \dots$
b) $-8, -2, \dots$

Summary

Arithmetic progressions

- An arithmetic progression (AP) is a sequence in which each term is formed by adding a constant amount to the previous term.
- The formula for the n th term of an AP with first term a and common difference d is: $T_n = a + (n - 1)d$.
- If a, b and c are three consecutive terms in an AP, then b is the arithmetic mean of a and c .
- An arithmetic series is the sum of an arithmetic progression.
- The sum of the first n terms of an AP is: $S_n = \frac{n}{2}[2a + (n - 1)d]$.
- The formula for the sum of the first n terms in an AP, where l is the last term, is: $S_n = \frac{n}{2}[a + l]$.

Geometric progressions

- A geometric progression (GP) is a sequence in which each term is formed by multiplying the previous term by a constant amount.
- The formula for the n th term of a GP with first term a and common ratio r is: $T_n = ar^{n-1}$.
- If a, b and c are three consecutive terms in a GP, then b is the geometric mean of a and c .
- A geometric series is the sum of a geometric progression.
- The sum of the first n terms in a GP is: $S_n = \frac{a(1 - r^n)}{(1 - r)}$, or $S_n = \frac{a(r^n - 1)}{(r - 1)}$.
- The sum to infinity of a GP is: $S_\infty = \frac{a}{(1 - r)}$ if and only if $-1 < r < 1$.

Revision exercises (remedial)

- Say whether each of the following is an AP, an GP, or neither.

a) 1, 4, 9, 16, ...	b) 125, 25, 5, 1, ...
c) 1.7, 2, 2.3, 2.6, ...	d) 1, 22, 333, 4444, ...
e) -5, -11, -17, -23, ...	f) 640, -320, 160, -80, ...

 $(6 \times 1 = 6)$
- For each of the APs in Question 1, write down the value of a and d . (4)
- For each of the GPs in Question 1, write down the value of a and r . (4)

Revision exercises

- For each of the following APs, find a formula for the n th term.

a) 0.5, 4, 7.5, ...	$(2 \times 3 = 6)$
b) -8, -29, -50, ...	

Revision and assessment continued

5 For each of the following GPs, find a formula for the n th term.

- $-3, 12, -48, \dots$ $(2 \times 3 = 6)$
- $50, 10, 2, \dots$ $(2 \times 3 = 6)$

6 Calculate S_{100} of an AP with $T_5 = 5$ and $T_{18} = 99$. (7)

7 The sum of the first five terms of an AP is -15 and the sum of the next six terms is 279 . Calculate the sum of the first 50 terms. (7)

8 Form an AP that has five arithmetic means between 102 and 36 . (5)

9 Insert three geometric means between 4 and 324 . Find all the possibilities. (5)

10 Calculate the sum to infinity of a GP with $T_2 = \frac{1}{12}$ and $T_5 = \frac{2}{81}$. (7)

11 A GP is such that $r = -\frac{4}{5}$ and $S_{\infty} = \frac{125}{9}$. Find the first term of the GP. (3)

Total marks: 60

Assessment exercises

1 Find the first term in the sequence for which T_{68} is -587 and $d = -9$. (2)

2 Calculate T_{17} of an AP with $T_{10} = 41$ and $T_{25} = 101$. (7)

3 Calculate the sum of the first ten terms of the GP: $1, 6, 36, \dots$ (2)

4 Find the common ratio of a GP with $S_{\infty} = \frac{2}{3}$ and $T_1 = 1$. (3)

5 The first three terms of a GP are $x - 2, x + 1$ and $x + 7$. Find:

- the value of x
- the common ratio
- S_6

$(3 \times 3 = 9)$

6 120 000 people visited the Lusaka Agricultural Show on its opening day. Thereafter, the attendance fell each day by 10% of the number on the previous day.

- If the show closed after seven days, how many people visited it? (2)
- If the show had been kept open indefinitely, calculate the maximum number of people who could have visited the show. (2)

7 In recent years, Zambia has recorded a steady decline in the spread of HIV and AIDS. In 2002, 16.1% of the adult population of Zambia was infected, but in 2007, this figure had dropped to 14.3% . Assume that the percentage of adults infected each year formed a GP.

- Calculate the value of r for the five years from 2002 to 2007. Write your answer as a decimal fraction, correct to eight decimal places. (4)
- If r remained constant over the next five years, what percentage of the adult population was infected in 2012? Write your answer correct to one decimal place. (2)

8 A GP has $S_{\infty} = 36$ and $S_3 = \frac{81}{2}$. Find the first three terms of the GP. (7)

Total marks: 40

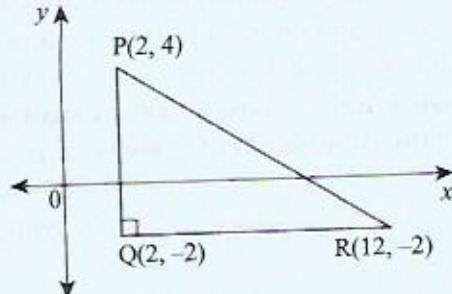


Sub-topics	Specific Outcomes
The length of a straight line between two points	<ul style="list-style-type: none"> Calculate the length of a straight line
The midpoint of two points	<ul style="list-style-type: none"> Calculate the midpoint of two points
The gradient of a line segment	<ul style="list-style-type: none"> Calculate the gradient of a line segment
The equation of a straight line	<ul style="list-style-type: none"> Find the equation of a straight line
Parallel and perpendicular lines	<ul style="list-style-type: none"> Find the gradients of parallel and perpendicular lines Use gradients of parallel and perpendicular lines to find equations

Starter activity

Work in pairs for this activity.

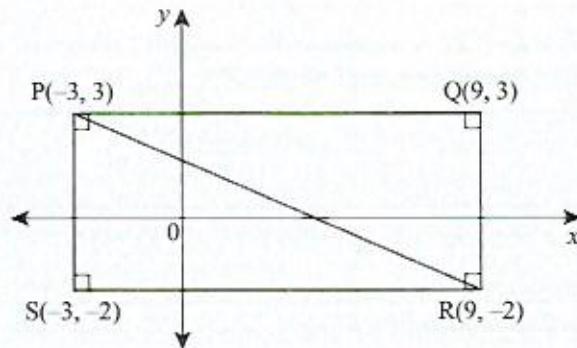
- Consider the formula $c^2 = a^2 + b^2$, where a and b are the two short sides of a right-angled triangle and c is the hypotenuse of the triangle.
 - What do we call this formula?
 - How do you think you can use this formula to solve problems in real life?
- A right-angled triangle has two short sides that measure 9 units and 12 units respectively. Calculate the length of the hypotenuse.
- Discuss $\triangle PQR$ in the diagram alongside with your partner. How could you use the formula in Question 1 above to find the length of PR ?



The length of a straight line is the distance between the two endpoints of the line.

Worked example 1

In the diagram below, PQRS is a rectangle on the Cartesian plane. Calculate the length of diagonal PR.



Answer

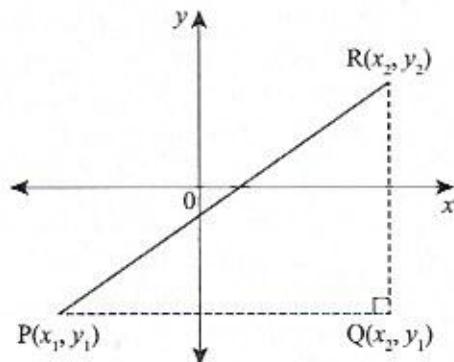
The distance $PQ = 9 - (-3) = 12$ units and $QR = 3 - (-2) = 5$ units.

To find the length of the diagonal PR, we use the Theorem of Pythagoras, as follows:

$$\begin{aligned} PR^2 &= PS^2 + SR^2 \\ &= 5^2 + 12^2 \\ &= 25 + 144 \\ &= 169 \\ \therefore PR &= \sqrt{169} = 13 \text{ units} \end{aligned}$$

We can now derive a general formula that can be used to calculate the length of a straight line between two points.

In the diagram on the right, $P(x_1, y_1)$ and $R(x_2, y_2)$ are two points on the Cartesian plane. Point Q has been constructed such that PQ is parallel to the x -axis and QR is parallel to the y -axis. ΔPQR is a right angle, and the coordinates of Q are (x_2, y_1) .



We can see that $PQ = x_2 - x_1$ and $RQ = y_2 - y_1$.
To find the length of the diagonal PR, we use the Theorem of Pythagoras, as follows:

$$\begin{aligned} PR^2 &= PQ^2 + RQ^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ \therefore PR &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \end{aligned}$$

So, the distance, d , between any two points on the Cartesian plane with coordinates (x_1, y_1) and (x_2, y_2) is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Worked example 2

Find the length of each of the following lines on the Cartesian plane.

- AB, if A(9, 5) and B(3, 13)
- JK, if J(-5, 4) and K(-2, -7)

Answers

$$\begin{aligned} \text{a) } AB &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} & \text{b) } JK &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(3 - 9)^2 + (13 - 5)^2} & &= \sqrt{(-2 - (-5))^2 + (-7 - 4)^2} \\ &= \sqrt{(-6)^2 + 8^2} & &= \sqrt{(-2 + 5)^2 + (-11)^2} \\ &= \sqrt{36 + 64} & &= \sqrt{3^2 + (-11)^2} \\ &= \sqrt{100} & &= \sqrt{9 + 121} \\ &= 10 \text{ units} & &= \sqrt{130} \text{ units} \end{aligned}$$

Note

It does not matter which point you choose to be point 1, and which point you choose to be point 2. The answers will be the same. You can leave your final answer in surd form, where necessary.

Activity 1

- Calculate the length of AB in Example a) above, but this time use A(9, 5) as (x_2, y_2) and B(3, 13) as (x_1, y_1) . Do you still get $AB = 10$ units?
If not, check your calculation carefully.
 - Calculate the length of JK in Example b) above, but this time use J(-5, 4) as (x_2, y_2) and K(-2, -7) as (x_1, y_1) . Do you still get $JK = \sqrt{130}$ units?
If not, check your calculation carefully.
- Find the length of each of the following lines on the Cartesian plane.
 - CD, if C(2, 2) and D(4, 3)
 - ST, if S(-3, -3) and T(-2, 3)
 - FG, if F(-10, 5) and G(3, 4)

The **midpoint** of a line segment is the point on the line segment that lies exactly halfway between the two endpoints of the line segment.

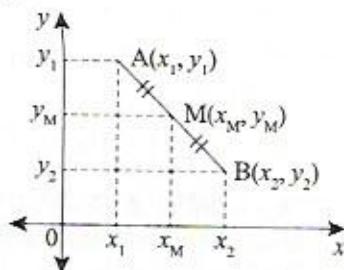
In the diagram alongside, $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points on the Cartesian plane. $M(x_M, y_M)$ is the midpoint of AB . x_M is halfway between x_1 and x_2 , so $x_M = \frac{x_1 + x_2}{2}$.

Similarly, y_M is halfway between y_1 and y_2 , so $y_M = \frac{y_1 + y_2}{2}$.

The coordinates of M are therefore $(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})$.

So, the coordinates of the midpoint of (x_1, y_1) and (x_2, y_2) are:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).$$



Note

- The x -coordinate of the midpoint of two points is the average of the x -coordinates of the points.
- The y -coordinate of the midpoint of two points is the average of the y -coordinates of the points.

New word

midpoint (of a line segment): the point on the line segment that lies exactly halfway between the two endpoints of the line segment

Worked example 3

- Find the coordinates of the midpoint of:
 - $(6, 3)$ and $(4, 5)$
 - $(-3, 4)$ and $(-9, 4)$.
- Find the coordinates of P , if the midpoint of PQ is $(2, -4)$ and point Q is $(6, -8)$.

Answers

$$\begin{aligned} 1 \text{ a) Midpoint} &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \\ &= \left(\frac{6+4}{2}, \frac{3+5}{2} \right) \\ &= \left(\frac{10}{2}, \frac{8}{2} \right) \\ &= (5, 4) \end{aligned}$$

$$\begin{aligned} \text{b) Midpoint} &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \\ &= \left(\frac{-3-9}{2}, \frac{4+4}{2} \right) \\ &= \left(\frac{-12}{2}, \frac{8}{2} \right) \\ &= (-6, 4) \end{aligned}$$

Worked example 3 (continued)

$$2 \frac{x_p + x_Q}{2} = x_M$$

$$\therefore \frac{x_p + 6}{2} = 2$$

$$\therefore x_p + 6 = 4$$

$$\therefore x_p = -2$$

$$\frac{y_p + y_Q}{2} = y_M$$

$$\therefore \frac{y_p - 8}{2} = -4$$

$$\therefore y_p - 8 = -8$$

$$\therefore y_p = 0$$

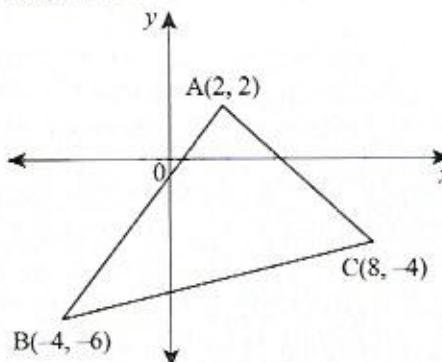
So, the coordinates of P are $(-2, 0)$.

Activity 2

1 Find the coordinates of the midpoint of:

a) $(2, 2)$ and $(4, 6)$ b) $(-5, 3)$ and $(-1, 9)$
 c) $(0, 6)$ and $(-6, 0)$ d) (a, b) and $(2a, 4b)$.

2 The vertices of $\triangle ABC$ are $A(2, 2)$, $B(-4, -6)$ and $C(8, -4)$, as shown in the diagram below.



Calculate the coordinates of:

a) M, the midpoint of BC
 b) N, the midpoint of AM.
 3 S $(5, 2)$ is the midpoint of R $(a + 2, 5)$ and T $(6, b)$. Calculate the values of a and b .
 4 The coordinates of A and B are $(-5, 1)$ and $(-7, -8)$ respectively.
 a) Find the coordinates of C, if B is the midpoint of AC.
 b) Find the coordinates of D, if C is the midpoint of AD.
 5 A line segment AE with coordinates A $(6, -6)$ and E $(14, 2)$ is divided into four equal segments by points B, C and D. Calculate the coordinates of:
 a) C b) B c) D.

The gradient-intercept form

The standard form of the equation of a straight line is $y = mx + c$, where m is the gradient and c is the y -intercept. The y -intercept is the y -coordinate at the point where the line cuts the y -axis. Similarly, the x -intercept is the x -coordinate at the point where the line cuts the x -axis.

If we have the gradient of a line, as well as the y -intercept, then we have the equation of the line.

Note

On the Cartesian plane, $y = 0$ on the x -axis and $x = 0$ on the y -axis.

New words

standard form (of the equation of a straight line): $y = mx + c$, where m is the gradient and c is the y -intercept

y -intercept: the y -coordinate at the point where a line cuts the y -axis

x -intercept: the x -coordinate at the point where a line cuts the x -axis

Worked example 6

Find the equation of the straight line:

- with a gradient of 2 and a y -intercept of 5
- with a gradient of $-\frac{5}{4}$ and a y -intercept of 0
- with a gradient of $\frac{7}{8}$ and a y -intercept of -3.

Answers

a) $m = 2$ and $c = 5$	b) $m = -\frac{5}{4}$ and $c = 0$	c) $m = \frac{7}{8}$ and $c = -3$
$y = mx + c$	$y = mx + c$	$y = mx + c$
$\therefore y = 2x + 5$	$\therefore y = -\frac{5}{4}x + 0$	$\therefore y = \frac{7}{8}x - 3$
	$\therefore y = -\frac{5}{4}x$	

If we have the gradient of a line, as well as the x -intercept, then we have to calculate the value of c .

The straight line $y = mx + c$ crosses the x -axis at the point where $y = 0$. So, we substitute the point (x -intercept, 0) into the equation $y = mx + c$ to find the value of c .

Worked example 7

Find the equation of the straight line:

- with a gradient of 2 and an x -intercept of 4
- with a gradient of $-\frac{3}{5}$ and an x -intercept of 5
- with a gradient of $\frac{4}{7}$ and an x -intercept of -9.

Worked example 7 (continued)

Answers

a) $y = mx + c$

$\therefore y = 2x + c$

Substitute in the point (4, 0):

$\therefore 0 = 2(4) + c$

$\therefore c = -8$

$\therefore y = 2x - 8$

b) $y = mx + c$

$\therefore y = -\frac{3}{5}x + c$

Substitute in the point (5, 0):

$\therefore 0 = -\frac{3}{5}(5) + c$

$\therefore c = 3$

$\therefore y = -\frac{3}{5}x + 3$

c) $y = mx + c$

$\therefore y = \frac{4}{7}x + c$

Substitute in the point (-9, 0):

$\therefore 0 = \frac{4}{7}(-9) + c$

$\therefore c = \frac{36}{7}$

$\therefore y = \frac{4}{7}x + \frac{36}{7}$

Activity 5

- Find the equation of the straight line with a y -intercept of -1 and
 - a gradient of 3
 - a gradient of -7.
- Find the equation of the straight line with a gradient of 4 and:
 - a y -intercept of 6
 - an x -intercept of 6.
- Find the equation of the straight line with a gradient of $-\frac{1}{2}$ and:
 - a y -intercept of $\frac{3}{2}$
 - an x -intercept of $\frac{3}{2}$
- Find the equation of the straight line with a gradient of 2.5 and:
 - an x -intercept of 5
 - an x -intercept of -7.
- Find the equation of the straight line with an x -intercept of 12 and
 - a gradient of $\frac{1}{3}$
 - a gradient of $-\frac{1}{4}$
 - a gradient of -2.

The double-intercept form

If we have both the x - and the y -intercepts of a straight line, then we have to calculate the gradient.

The straight line $y = mx + c$ crosses the x -axis at the point where $y = 0$. So, we substitute the point (x -intercept, 0) into the equation $y = mx + c$ to find the value of m . Alternatively, we can use the formula $m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$ to calculate the gradient.

Worked example 8

Find the equation of the straight line:

- with an x -intercept of 3 and a y -intercept of 6
- with an x -intercept of -12.5 and a y -intercept of 5.

Answers

a) Method 1:

$c = 6$ and the line passes through the point $(3, 0)$

$$y = mx + c$$
$$\therefore y = mx + 6$$

Substitute in the point $(3, 0)$:

$$\therefore 0 = m(3) + 6$$
$$\therefore m = -2$$
$$\therefore y = -2x + 6$$

Method 2:

$c = 6$ and the line passes through the points $(3, 0)$ and $(0, 6)$

$$m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$
$$= \frac{(6 - 0)}{(0 - 3)}$$
$$= \frac{6}{-3}$$
$$= -2$$
$$\therefore y = -2x + 6$$

b) Method 1:

$c = 5$ and the line passes through the point $(-12.5, 0)$

$$y = mx + c$$
$$\therefore y = mx + 5$$

Substitute in the point $(-12.5, 0)$:

$$\therefore 0 = m(-12.5) + 5$$
$$\therefore m = \frac{5}{12.5} = \frac{2}{5}$$
$$\therefore y = \frac{2}{5}x + 5$$

Method 2:

$c = 5$ and the line passes through the points $(-12.5, 0)$ and $(0, 5)$

$$m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$
$$= \frac{(5 - 0)}{(0 - (-12.5))}$$
$$= \frac{5}{12.5}$$
$$= \frac{2}{5}$$
$$\therefore y = \frac{2}{5}x + 5$$

Activity 6

Work in pairs for this activity.

- Discuss the examples above. Which method does each of you prefer?
- Use any method that you like to find the equation of the straight line:
 - with an x -intercept of -3 and a y -intercept of 3
 - with an x -intercept of -5 and a y -intercept of -10
 - with an x -intercept of 4 and a y -intercept of -5
 - with an x -intercept of -4 and a y -intercept of 2
 - with an x -intercept of 6.5 and a y -intercept of 3.5 .

SUB-TOPIC 5

Parallel and perpendicular lines

If two straight lines are parallel, then their gradients are the same. So, if m_1 and m_2 are the gradients of two parallel lines, then $m_1 = m_2$.

If two straight lines are perpendicular, then the product of their gradients is negative one. So, if m_1 and m_2 are the gradients of perpendicular lines, then $m_1 \times m_2 = -1$.

Worked example 9

Find the gradient of a straight line that is a) parallel and b) perpendicular to the lines given by each of the following equations:

1 $y = x + 3$

2 $y = -\frac{3}{7}x$

3 $2x + 5y = 10$

4 $6x - 2y + 7 = 0$

Answers

1 a) $m = 1$

b) $m_1 \times m_2 = -1$

$\therefore m \times 1 = -1$

$\therefore m = -1$

3 a) $2x + 5y = 10$

$\therefore y = -\frac{2}{5}x + 2$

$\therefore m = -\frac{2}{5}$

b) $m_1 \times m_2 = -1$

$\therefore m \times \left(-\frac{2}{5}\right) = -1$

$\therefore m = \frac{5}{2}$

2 a) $m = -\frac{3}{7}$

b) $m_1 \times m_2 = -1$

$\therefore m \times \left(-\frac{3}{7}\right) = -1$

$\therefore m = \frac{7}{3}$

4 a) $6x - 2y + 7 = 0$

$\therefore y = 3x + \frac{7}{2}$

$\therefore m = 3$

b) $m_1 \times m_2 = -1$

$\therefore m \times 3 = -1$

$\therefore m = -\frac{1}{3}$

Activity 7

Work in pairs for this activity.

Find the gradient of a straight line that is a) parallel and b) perpendicular to the lines given by each of the following equations:

1 $y = 4x + 10$

2 $y = -\frac{6}{11}x - \frac{1}{3}$

3 $y = -2x + 1$

4 $3x + 4y = 12$

5 $-x - 5y + 8 = 0$

6 $y = 2(4x + 1)$

7 $6y + 4 = -3x - 2$

8 $x = 9y + 18$

You will now use your knowledge of parallel and perpendicular lines to find the equations of straight lines.

Worked example 10

- 1 Find the equation of the straight line that passes through the point (4, 6) and is parallel to the line $y = 3x - 8$.
- 2 Find the equation of the straight line that is perpendicular to the line $3x - 2y = 4$ and passes through the point (-9, 5).

Answers

1 The gradient of our new line is 3, because the line is parallel to the line $y = 3x - 8$.

$$y = mx + c$$

$$\therefore y = 3x + c$$

Substitute in the point (4, 6):

$$\therefore 6 = 3(4) + c$$

$$\therefore c = -6$$

$$\therefore y = 3x - 6$$

2 $3x - 2y = 4$
 $\therefore 2y = 3x - 4$
 $\therefore y = \frac{3}{2}x - \frac{4}{3}$

If the gradient of our new line is m , then:

$$m \times \frac{3}{2} = -1$$
$$\therefore m = -\frac{2}{3}$$

$$y = mx + c$$
$$\therefore y = -\frac{2}{3}x + c$$

Substitute in the point (-9, 5):

$$\therefore 5 = -\frac{2}{3}(-9) + c$$
$$\therefore 5 = 6 + c$$
$$\therefore c = -1$$
$$\therefore y = -\frac{2}{3}x - 1$$

Activity 8

- 1 Find the equation of the straight line that passes through the point (3, 9) and is a) parallel and b) perpendicular to the line $y = x - 5$.
- 2 Find the equation of the straight line that is (i) parallel and (ii) perpendicular to:
 - $4x + 3y = 1$, and passes through the point (0, -1)
 - $x + 2y = -2$, and passes through the point (-1, 2).
- 3 A line is drawn through the point (3, 2), parallel to the line $y + 3x = 4$.
 - Find the equation of the line.
 - Find the equation of the perpendicular line that passes through the same point.

Summary

The length of a straight line between two points

- The length of a straight line is the distance between the two endpoints of the line.
- The distance, d , between any two points on the Cartesian plane with coordinates (x_1, y_1) and (x_2, y_2) is: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

The midpoint of two points

- The midpoint of a line segment is the point on the line segment that lies exactly halfway between the two endpoints of the line segment.
- The coordinates of the midpoint of (x_1, y_1) and (x_2, y_2) are: $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

The gradient of a line segment

- The gradient of a straight line is the steepness, or slope of the line.
- If the line slopes to the right (/), then the gradient is positive. If the line slopes to the left (\), then the gradient is negative.
- The gradient, m , of the line through any two points on the Cartesian plane with coordinates (x_1, y_1) and (x_2, y_2) is: $m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$.
- A horizontal line has gradient of zero.
- A vertical line has an infinite (or undefined) gradient.

The equation of a straight line

- If we have the gradient, m , of a line as well as one point (x_1, y_1) that lies on the line, then the equation of the line is: $y - y_1 = m(x - x_1)$.
- If we have two points, (x_1, y_1) and (x_2, y_2) , then the equation of the line that passes through these points is: $y - y_1 = \frac{(y_2 - y_1)}{(x_2 - x_1)}(x - x_1)$.
- The standard form of the equation of a straight line is $y = mx + c$, where m is the gradient and c is the y -intercept.
- The y -intercept is the y -coordinate at the point where the line cuts the y -axis.
- The x -intercept is the x -coordinate at the point where the line cuts the x -axis.

Parallel and perpendicular lines

- If two straight lines are parallel, then their gradients are the same. So, if m_1 and m_2 are the gradients of two parallel lines, then $m_1 = m_2$.
- If two straight lines are perpendicular, then the product of their gradients is negative one. So, if m_1 and m_2 are the gradients of perpendicular lines, then $m_1 \times m_2 = -1$.

Revision exercises (remedial)

- Calculate the distance of point $P(3, 4)$ from the origin. (3)
- Line AB has the equation $y = \frac{1}{4}x + 6$. Write down the gradient of a line that is:
 - parallel to line AB
 - perpendicular to line AB. $(2 \times 1 = 2)$

Revision exercises

- $P(2, 4)$ and $Q(6, 10)$ are two points on the Cartesian plane.
 - Find the length of PQ. (3)
 - Find the coordinates of the midpoint of PQ. (3)
 - Find the gradient of PQ. (3)
 - Find the equation of PQ. (3)
- Find the equation of each of the following lines.
 - The line that passes through $(1, 5)$ and has a gradient of -2 . (3)
 - The line that cuts the x -axis at $(6, 0)$ and the y -axis at $(0, -8)$. (3)
 - The line that passes through the point $(-4, 2)$ and is parallel to the line $2y + x - 4 = 0$. (3)
 - The line that cuts the x -axis at $(-3, 0)$ and is perpendicular to the line $x - y = 5$. (3)
- $E(-1, 1)$ is the midpoint of $D(a - 2, 5)$ and $F(2, b)$. Calculate the values of a and b . (6)

Total marks: 25

Assessment exercises

- The vertices of $\triangle ABC$ are $A(-5, 2)$, $B(3, 4)$ and $C(-1, -1)$, as shown in the diagram alongside.
 - Find the coordinates of P, Q and R, the midpoints of AB, BC and AC respectively. (9)
 - Show that:
 - $PQ \parallel AC$
 - $QR \parallel AB$
 - $PR \parallel BC$
 - Show that:
 - $PQ = \frac{1}{2}AC$
 - $QR = \frac{1}{2}AB$
 - $PR = \frac{1}{2}BC$ $(3 \times 7 = 21)$
- $PQRS$ is a square on the Cartesian plane. P and Q have coordinates $(-1, -5)$ and $(3, -5)$ respectively. Find all the possible coordinates of R and S. (7)
- A line segment AF with coordinates $A(-8, 13.5)$ and $F(2, -1.5)$ is divided into five equal segments by points B, C, D and E. Calculate the coordinates of B, C, D and E. (8)

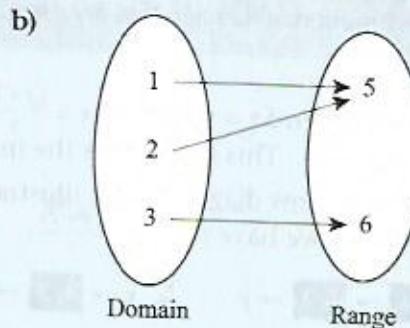
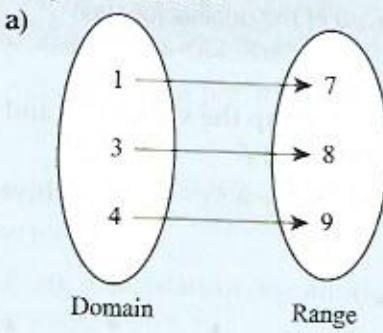
Total marks: 60

Sub-topics	Specific Outcomes
Inverse functions	<ul style="list-style-type: none"> Find inverses of one-to-one functions
Composite functions	<ul style="list-style-type: none"> Simplify composite functions
Application	<ul style="list-style-type: none"> Solve problems involving linear functions

Starter activity

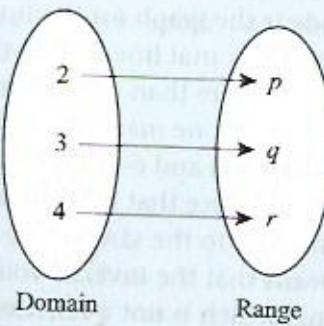
Discuss the questions below as a class.

- 1 Compare and contrast the meanings of the words relation and function.
- 2 Explain why all functions are relations, but not all relations are functions.
- 3 State whether each function below is a one-to-one mapping. Give reasons for your answers.



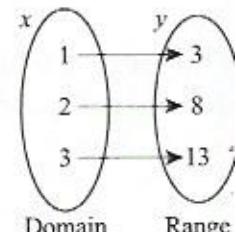
4 A mapping of some elements of the domain and range of the function $f(x) = 2 + x$ is shown in the diagram on the right.

- a) Find the values of the elements p , q and r .
- b) Write a formula for a function which will map the elements of the range back onto the domain.



Find inverses of one-to-one functions

The diagram alongside shows a mapping of the domain $x = \{1, 2, 3\}$ under the function $f: x \rightarrow 5x - 2$ onto the range $y = \{3, 8, 13\}$. These elements can be written as a set of ordered pairs as follows: $f = \{(1, 3), (2, 8), (3, 13)\}$. The inverse function of f is that function that will reverse the order of the coordinates of these ordered pairs. The ordered pairs of the inverse function will therefore be $\{(3, 1), (8, 2), (13, 3)\}$. We want to find a function which will map the elements in the range back into the domain.



New words

mapping: the matching of the elements of one set to the elements of another set by means of a rule

domain: the set of x -values for which a function is defined

function: a one-to-one or a many-to-one mapping

range: the set of y -values for which a function is defined

ordered pairs: coordinate pairs of the form (x, y)

inverse function: a function that reverses the mapping of the original function

If $y = 5x - 2$, then $5x = y + 2$ and $x = \frac{(y+2)}{5}$. When we swap the variables x and y , we get $y = \frac{(x+2)}{5}$. This function is the inverse function of f .

Below are the flow diagrams that illustrate the function $y = 5x - 2$ and its inverse: $y = \frac{(x+2)}{5}$. So, we have $f^{-1}: x \rightarrow \frac{(x+2)}{5}$.

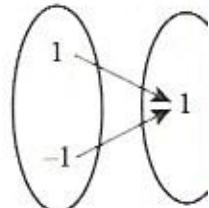
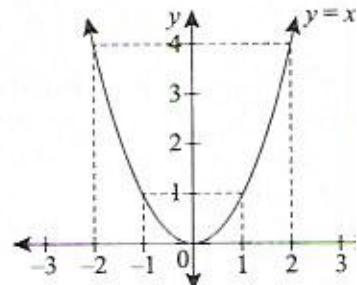
$$x \rightarrow \boxed{x \times 5} \rightarrow \boxed{-2} \rightarrow y \qquad x \rightarrow \boxed{+2} \rightarrow \boxed{\div 5} \rightarrow y$$

Not all functions have an inverse function.

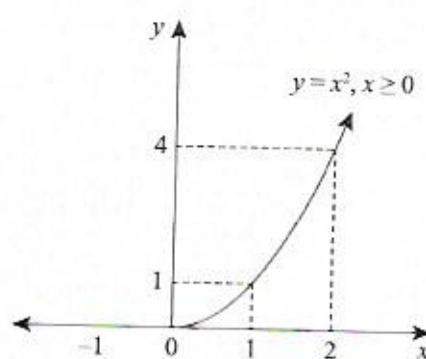
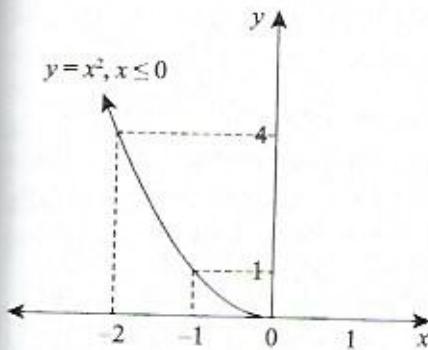
Alongside is the graph for the function $f: x \rightarrow x^2$.

If any horizontal line crosses the graph of a function in more than one point, then the graph is not a one-to-one mapping. In the diagram alongside, $(1, 1)$ and $(-1, 1)$ are ordered pairs of the function f . Notice that two arrows from the domain point to the same value (1) in the range.

This means that the inverse would be a one-to-many mapping, which is not a function. So, the function $f: x \rightarrow x^2$ does not have an inverse function. The inverse of a function exists if, and only if, the original function is a one-to-one mapping.



We can, however, restrict the domain of a function in such a way that its inverse exists. For example, if we restrict the domain of the function $f: x \rightarrow x^2$ as follows: $x \leq 0$, or $x \geq 0$, then we get the functions below, each of which has an inverse function.



The inverse of $f: x \rightarrow x^2; x \leq 0$ is $f^{-1}: x \rightarrow -\sqrt{x}$ and the inverse of $f: x \rightarrow x^2; x \geq 0$ is $f^{-1}: x \rightarrow \sqrt{x}$.

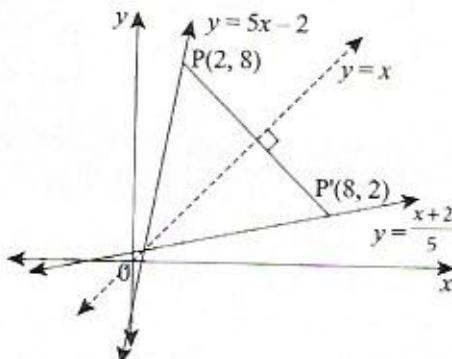
The graphs of a function f and its inverse function f^{-1}

On the right is a diagram of the graphs of the function $f: x \rightarrow 5x - 2$ and its inverse $f^{-1}: \frac{(x+2)}{5}$, as well as the line $y = x$.

$f(2) = 5(2) - 2 = 8$, so $P(2, 8)$ lies on the line $y = 5x - 2$.

$f^{-1}(8) = \frac{(8+2)}{5} = 12$, so $P'(8, 2)$, lies on the line $y = \frac{(x+2)}{5}$.

P and P' are reflections of one another in the line $y = x$.



If we choose any other value of x , we will find that the coordinates of the point $(x, f(x))$ are swapped around in the image on f^{-1} . So, the graphs of lines $f(x)$ and $f^{-1}(x)$ are reflections of each other in the line $y = x$.

Worked example 1

- Find the inverse function of $f: x \rightarrow 3x - 5$.
- Given $f: x \rightarrow \frac{(x-1)}{2}$, calculate:
 - $f^{-1}(x)$
 - $f^{-1}(-3)$.
- Find the inverse of $f: x \rightarrow 5 - x$.

Worked example 1 (continued)

Answers

1 $f: x \rightarrow 3x - 5$

$\therefore y = 3x - 5$

$\therefore 3x = y + 5$

$\therefore x = \frac{y+5}{3}$

$\therefore f^{-1}(x) = \frac{(x+5)}{3}$

$\therefore f^{-1}: x \rightarrow \frac{(x+5)}{3}$

2 a) $f: x \rightarrow \frac{(x-1)}{2}$

$\therefore y = \frac{(x-1)}{2}$

$\therefore 2y = x - 1$

$\therefore x = 2y + 1$

$\therefore f^{-1}(x) = 2x + 1$

$\therefore f^{-1}: x \rightarrow 2x + 1$

b) $f^{-1}(-3) = 2(-3) + 1 = -5$

3 $f: x \rightarrow 5 - x$

$\therefore y = 5 - x$

$\therefore x = 5 - y$

$\therefore f^{-1}(x) = 5 - x$

$\therefore f^{-1}: x \rightarrow 5 - x$

Write the function in standard form.

Make x the subject of the formula.

Swap the variables x and y to obtain the inverse function.

Write the function in standard form.

Swap the variables x and y to obtain the inverse function.

Write the function in standard form.

Swap the variables x and y to obtain the inverse function.

New word

self-inverse function: a function that is identical to its inverse

Activity 1

1 Find the inverses of the following functions. Identify any self-inverse functions.

a) $f: x \rightarrow 5x$

b) $f: x \rightarrow x + 3$

c) $f: x \rightarrow x$

d) $f: x \rightarrow 7 - x$

e) $f: x \rightarrow \frac{6}{x}; x \neq 0$

f) $g: x \rightarrow \frac{5}{(x+2)}; x \neq -2$

g) $h: x \rightarrow \frac{(3x+2)}{4}$

h) $g: x \rightarrow \frac{3}{x}; x \neq 0$

i) $h: x \rightarrow \frac{x}{4} - 1$

j) $f: x \rightarrow 5 - 3x$

2 If $f^{-1}: x \rightarrow 2x + 5$, find f .

3 Find g , given that its inverse function is $g^{-1}: x \rightarrow \frac{(x+3)}{5}$.

4 Find the inverse function of each of these functions, where the inverse function exists. If it does not exist, explain why this is the case.

a) $f: x \rightarrow 2x + 2$

b) $f: x \rightarrow x^2 - 1$

SUB-TOPIC 2

Composite functions

Consider the functions $f: x \rightarrow 2x - 3$ and $g: x \rightarrow x + 2$. The image of 4 under the function $f(x) = 2x - 3$ is $2(4) - 3 = 5$, so $f(4) = 5$. If we now let the function $g(x) = x + 2$ act on $f(4)$, we obtain $g[f(4)]$. Given that $f(4) = 5$, we can write $g[f(4)] = g(5) = 5 + 2 = 7$. Thus 4 has been mapped onto 7 by f , followed by g .

We can find a single function h which combines f and g . Thus we have $h(x) = g[f(x)]$, or simply $h = gf$. Note that the first function is always written on the right. In this case, $h(x) = 2x - 3$ and $g(x) = x + 2$ and so x is mapped onto $2x - 3$ by f , which gives us the starting value for g . Then $h(x) = g[f(x)] = (f(x)) + 2 = (2x - 3) + 2 = 2x - 1$. So, the new function is $h: 2x - 1$. This function is called the **composite function** of g and f . We write $h(x) = (g \circ f)(x)$, or simply $g \circ f$, which we read as "g of f".

Is $g[f(x)]$ the same as $f[g(x)]$?

$f[g(x)] = 2(g(x)) - 3 = 2(x + 2) - 3 = 2x + 1$. When we compare $g[f(x)] = 2x - 1$ and $f[g(x)] = 2x + 1$, we notice that the results are different.

$g[f(x)]$ is therefore not the same as $f[g(x)]$. We say that the combination of functions is not commutative, so the order in which they are written is very important and cannot be interchanged. In some cases, though, we do find that $g[f(x)]$ is equal to $f[g(x)]$, but this is not generally true.

If f and g are functions where $g: x \rightarrow y$ and $f: y \rightarrow z$, then their composite function, denoted by $f \circ g$, is the function $y \rightarrow z$, which is given by $f \circ g(x) = f[g(x)]$, for all values of x in the domain of g such that $g(x)$ is in the domain of f . That is, first evaluate g on x and then f on $g(x)$.

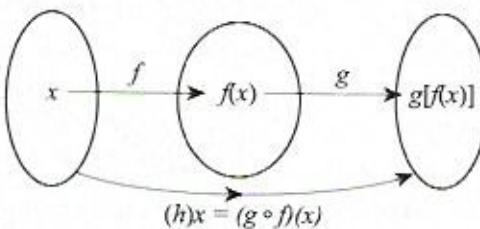
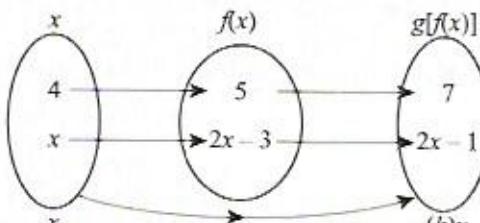
Worked example 2

1 If $f(x) = x^2$ and $g(x) = x - 2$, find:

- a) $f \circ g(x)$
- b) $g \circ f(x)$
- c) $f \circ g(3)$.

2 Given that $f: x \rightarrow 3x - 2$, find:

- a) $f \circ f^{-1}(x)$
- b) $f^{-1} \circ f(x)$.



New word

composite function: a function that is a combination of two or more functions

Composite functions

Consider the functions $f: x \rightarrow 2x - 3$ and $g: x \rightarrow x + 2$. The image of 4 under the function $f(x) = 2x - 3$ is $2(4) - 3 = 5$, so $f(4) = 5$. If we now let the function $g(x) = x + 2$ act on $f(4)$, we obtain $g[f(4)]$. Given that $f(4) = 5$, we can write $g[f(4)] = g(5) = 5 + 2 = 7$. Thus 4 has been mapped onto 7 by f , followed by g .

We can find a single function h which combines f and g . Thus we have $h(x) = g[f(x)]$, or simply $h = gf$. Note that the first function is always written on the right. In this case, $h(x) = 2x - 3$ and $g(x) = x + 2$ and so x is mapped onto $2x - 3$ by f , which gives us the starting value for g . Then $h(x) = g[f(x)] = (f(x)) + 2 = (2x - 3) + 2 = 2x - 1$. So, the new function is $h: 2x - 1$. This function is called the **composite function** of g and f . We write $h(x) = (g \circ f)(x)$, or simply $g \circ f$, which we read as "g of f".

Is $g[f(x)]$ the same as $f[g(x)]$?

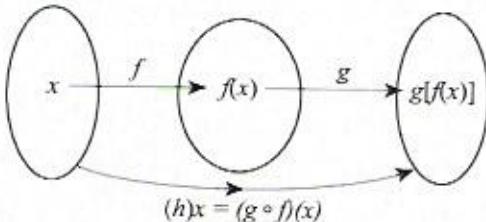
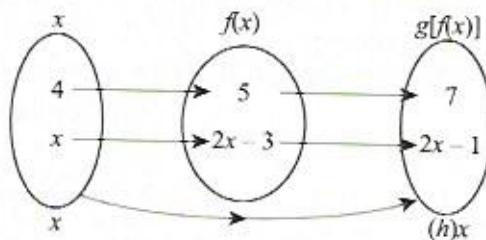
$f[g(x)] = 2(g(x)) - 3 = 2(x + 2) - 3 = 2x + 1$. When we compare $g[f(x)] = 2x - 1$ and $f[g(x)] = 2x + 1$, we notice that the results are different.

$g[f(x)]$ is therefore not the same as $f[g(x)]$. We say that the combination of functions is not commutative, so the order in which they are written is very important and cannot be interchanged. In some cases, though, we do find that $g[f(x)]$ is equal to $f[g(x)]$, but this is not generally true.

If f and g are functions where $g: x \rightarrow y$ and $f: y \rightarrow z$, then their composite function, denoted by $f \circ g$, is the function $y \rightarrow z$, which is given by $f \circ g(x) = f[g(x)]$, for all values of x in the domain of g such that $g(x)$ is in the domain of f . That is, first evaluate g on x and then f on $g(x)$.

Worked example 2

- If $f(x) = x^2$ and $g(x) = x - 2$, find:
 - $f \circ g(x)$
 - $g \circ f(x)$
 - $f \circ g(3)$.
- Given that $f: x \rightarrow 3x - 2$, find:
 - $f \circ f^{-1}(x)$
 - $f^{-1} \circ f(x)$.



New word

composite function: a function that is a combination of two or more functions

Worked example 2 (continued)

Answers

$$\begin{array}{lll} 1 \text{ a) } f \circ g(x) = f[g(x)] & \text{b) } g \circ f(x) = g[f(x)] & \text{c) } f \circ g(3) = f[g(3)] \\ & = f(x-2) & = g(x^2) \\ & = (x-2)^2 & = x^2-2 \\ & & = f(3-2) \\ & & = f(1) \\ & & = 1^2 \\ & & = 1 \end{array}$$

$$\begin{array}{lll} 2 \text{ We first find } f^{-1}(x): & \text{a) } f \circ f^{-1}(x) = f[f^{-1}(x)] & \text{b) } f^{-1} \circ f(x) = f^{-1}[f(x)] \\ f(x) = 3x-2 & = f\left(\frac{(x+2)}{3}\right) & = f^{-1}[3x-2] \\ \therefore y = 3x-2 & = 3\left(\frac{(x+2)}{3}\right) - 2 & = \frac{(3x-2)+2}{3} \\ \therefore 3x = y+2 & = x & = x \\ \therefore x = \frac{(y+2)}{3} & & \\ \therefore f^{-1}(x) = \frac{(x+2)}{3} & & \end{array}$$

Note that $f \circ f^{-1}(x) = f^{-1} \circ f(x) = x$. This leads to the following important fact:

Two functions, $f(x)$ and $f^{-1}(x)$, are inverse functions if and only if $f \circ f^{-1}(x) = f^{-1} \circ f(x) = x$.

Activity 2

- 1 For the functions $f: x \rightarrow x-5$ and $g: x \rightarrow x+3$, find each of the following:
 - $f \circ g(x)$
 - $g \circ f(x)$
 - $g \circ g(x)$
 - $f \circ f(x)$
- 2 If $f: x \rightarrow x-5$ and $g: x \rightarrow x^2$, find:
 - $f \circ g$
 - $g \circ f$
 - f^{-1}
 - $g \circ f^{-1}$
 - $f^{-1} \circ g$
- 3 For the functions $f: x \rightarrow 3x+4$ and $g: x \rightarrow 2x-3$, find each of the following:
 - $f^{-1}(x)$
 - $g^{-1}(x)$
 - $f \circ g^{-1}(x)$
 - $f^{-1} \circ g$
 - $f^{-1} \circ g^{-1}$
 - $g^{-1} \circ f^{-1}$
- 4 Show that the functions $f(x) = 5x-6$ and $g(x) = \frac{(x+6)}{5}$ are inverses of one another.
- 5 If $f: x \rightarrow 3x$ and $g: x \rightarrow x-4$, show that:
 - $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$
 - $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$
- 6 If $f: x \rightarrow x+3$ and $g: x \rightarrow x^2-1$, evaluate:
 - $g \circ f(-2)$
 - $f \circ g(3)$
 - $g \circ f^{-1}(3)$

Solve problems involving linear functions

In this sub-topic, we apply the definitions and concepts of functions to solve problems involving linear functions.

New word

linear function: a function of the form $f(x) = ax + b$

Worked example 3

- 1 A function $f \circ g$ is defined by $f \circ g: x \rightarrow 2x + 1$. If $g: x \rightarrow x + 3$, find function f .
- 2 It costs a bicycle factory K12 000 to produce 20 bicycles and K32 000 to produce 60 bicycles of the same make.
 - a) Find the linear cost function for this situation.
 - b) What is the fixed cost for this situation?
 - c) What is the cost per bicycle?

Answers

- 1 In this case, f is linear, because no squares or higher powers appear in $f \circ g$.

Let $f(x) = ax + b$.

$$\therefore f \circ g: x \rightarrow a(x + 3) + b = ax + 3a + b$$

$$\therefore ax + 3a + b = 2x + 1$$

$$\therefore a = 2$$

$$\therefore 3(2) + b = 1$$

$$\therefore b = -5$$

$$\therefore f: x \rightarrow 2x - 5$$

- 2 a) Let $C(x) = ax + b$.

$$C(20) = 12\ 000$$

$$\therefore 20a + b = 12\ 000 \quad \textcircled{1}$$

$$C(60) = 32\ 000$$

$$\therefore 60a + b = 32\ 000 \quad \textcircled{2}$$

Subtract $\textcircled{1}$ from $\textcircled{2}$:

$$\therefore 40a = 20\ 000$$

$$\therefore a = 500 \quad \textcircled{3}$$

Substitute $\textcircled{3}$ into $\textcircled{1}$:

$$\therefore 20(500) + b = 12\ 000$$

$$\therefore b = 2\ 000$$

$$\therefore C(x) = 500x + 2\ 000$$

- b) The fixed cost is K2 000.

- c) The cost per bicycle is K500.

Activity 3

- 1 A manufacturer of school bags has a monthly overhead fixed cost of K6 000. It also costs him K75.50 to make each bag.
 - a) Write an equation to represent his fixed cost.
 - b) Calculate his total cost if he makes 250 school bags in a given month.
- 2 Mrs Banda operates a restaurant. Her operations cost is K525 when the sales are K600 and K750 when the sales are K900.
 - a) Find the linear cost function for this situation.
 - b) What is her total cost of operations if her sales are K1 000?
- 3 A company which produces ready-to-drink packs of Maheu has a fixed overhead cost of K4 000 per week. In addition, each pack of Maheu costs K0.40 to produce.
 - a) Write an equation that represents the total cost, $C(x)$, of producing x packs of Maheu per week.
 - b) How many packs of Maheu were produced in a given week if the total cost was K7 600?
- 4 A biologist carried out research in the Bangweulu swamps of Zambia and discovered that there is a linear relationship between the total length of a certain species of tortoise and the length of its outstretched neck, where the neck length is between 50 mm and 100 mm. Some of his measurements are shown in the table below.

Neck length	60 mm	95 mm
Total length	429 mm	674 mm



Did you know?

Maheu is a popular non-alcoholic traditional Zambian drink that can be bought in small ready-to-drink packs. It is traditionally made from finger millet, but it can also be made from mealie meal. Maheu has been around for as long as the Zambian people.



- a) Write a linear function relating the total length, $f(x)$, to the length of the neck, x .
- b) What is the total length of a tortoise whose neck length is 82 mm?

- 5 With Top network, a 2-minute phone call between Ndola and Lundazi costs K7.90 and a 5-minute call costs K16.00.
 - a) Find the linear cost function for this situation.
 - b) How much will Mr Nyirenda pay for a 15-minute phone call from Ndola to his village in Lundazi if he uses Top network?

Summary

Inverse functions

- A mapping is the matching of the elements of one set to the elements of another set by means of a rule.
- A function is a one-to-one or a many-to-one mapping.
- We use these notations for a function: $f(x) = \dots$ or $f: x \rightarrow \dots$.
- The domain of a function is the set of x -values for which the function is defined.
- The range of a function is the set of y -values for which the function is defined.
- The inverse of a function is a function that reverses the mapping of the original function.
- We use these notations for an inverse function: $f^{-1}(x) = \dots$ or $f^{-1}: x \rightarrow \dots$.
- The inverse function of a function exists if, and only if, the original function is a one-to-one mapping.
- If a function is a many-to-one mapping, we can restrict the domain of the function in such a way that the inverse function exists. This restriction is often of the form $x \leq 0$, or $x \geq 0$.
- The graphs of $f(x)$ and $f^{-1}(x)$ are reflections of each other in the line $y = x$.
- A function that is identical to its inverse is called a self-inverse function.

Composite functions

- A composite function is a function that is a combination of two or more functions.
- We use these notations for a composite function: if $h(x)$ is a composite function of g and f , then $h(x) = (g \circ f)(x)$, or $h = g \circ f$.
- $g[f(x)] \neq f[g(x)]$, so the combination of functions is not commutative.

Application

- A linear function is a function of the form $f(x) = ax + b$.
- A linear cost function is a function of the form $C(x) = ax + b$, where a is the cost of making or selling a single item, x is the number of items made or sold and b is the fixed cost of production or sales.

Revision exercises (remedial)

- 1 If $f(x) = \frac{(x-2)}{3}$, find the value of $f(11)$. (1)
- 2 Show that the function $h: x \rightarrow 4 - x$ is a self-inverse function. (2)
- 3 If $f: x \rightarrow 3x + 1$ and $g: x \rightarrow 2x - 3$, find:
 - $f \circ g$
 - $g \circ f$
$$(2 \times 2 = 4)$$
- 4 In the cost equation $C(x) = ax + b$, which variable represents:
 - the fixed cost
 - the cost of producing one item? (2 \times 1 = 2)

Revision exercises

5 If $f(x) = \frac{(x+a)}{5}$, find the value of a if $f(4) = 2$. (2)

6 Find the inverse of $f: x \rightarrow 3x - 2$ and show that $f \circ f^{-1}(x) = f^{-1} \circ f(x) = x$. (5)

7 If $f: x \rightarrow \frac{3}{(x+5)}$; $x \neq -5$, find f^{-1} and state the values of x that must be excluded from the domain of f^{-1} . (2)

8 The tax on a net income of K16 000 is K3 320, while on a net income of K16 400, the tax is K3 408. The function used to calculate the tax is $f(x) = ax + b$, where $f(x)$ is the tax payable and x is the net income.

- Calculate the values of a and b . (4)
- Explain the value of b in terms of this context. (1)
- Calculate the tax payable on a net income of K20 000. (2)

Total marks: 2

Assessment exercises

1 If $f: x \rightarrow \frac{(x+m)}{(x+2)}$; $x \neq -2$, find the value of m if $f(-1) = 2$. (2)

2 If $f: x \rightarrow 3x + a$, find the value of a if $f^{-1}(-2) = -3$. (2)

3 If $g: x \rightarrow \frac{6}{(x+2)}$; $x \neq -2$, find the value of $g^{-1}(4)$. (2)

4 The functions f and g are defined by $f: x \rightarrow 3x^2 - 2$ and $g: x \rightarrow x + 3$.

Find:

- $f \circ f$
- $f \circ g^{-1}$

5 If $f: x \rightarrow 2x + 3$ and $g: x \rightarrow x - 5$, find the value(s) of x for which $f \circ g^{-1} = f^{-1} \circ g$. (2)

6 In Livingstone, the tourist capital of Zambia, the taxi fare for a 1 km trip is K170 and a 3 km trip costs K330.

- Form the equation for the linear function.
- How much will Miyoba pay for a 2.5 km trip from his hotel to the Victoria Falls?

7 Given $f: x \rightarrow \sqrt{x+1}$.

- Find f^{-1} .
- Explain why the restriction $x \geq 0$ must be applied to f^{-1} .

8 Given the function $h: x \rightarrow \frac{(ax-1)}{(x-b)}$; $x \neq b$ and that $h(4) = 7$ and $h(5) = 4\frac{1}{2}$. Find the values of a and b .

9 If $f \circ g: x \rightarrow 3x - 6$, find function g if $f: x \rightarrow x - 4$.

Total marks: 10



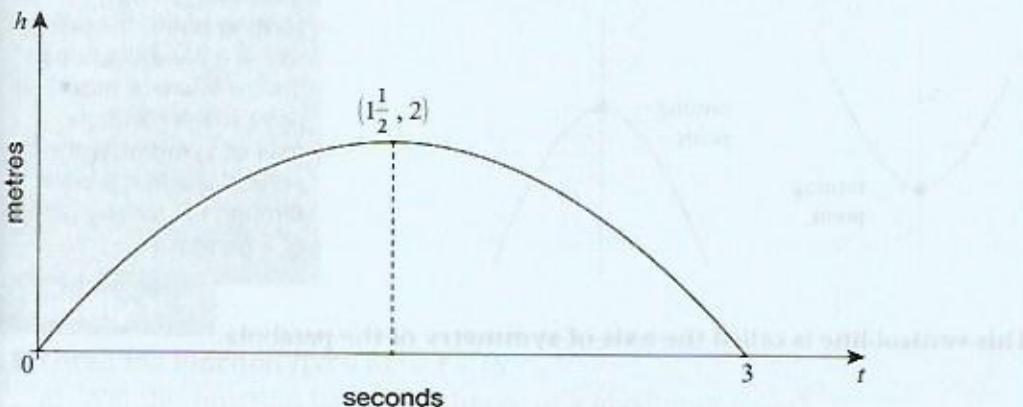
Sub-topics	Specific Outcomes
Introduction to quadratic functions	<ul style="list-style-type: none"> Explain the quadratic function and its graph Sketch the graph of a quadratic function

Starter activity

Work in pairs for this activity.

Study the diagram carefully, then answer the questions that follow.

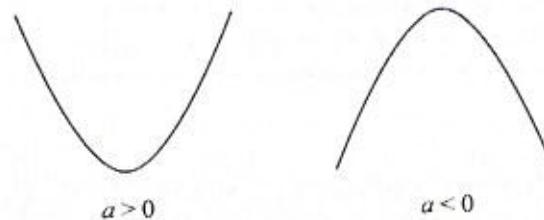
When a ball is thrown or kicked in the air, its path can be described by a quadratic function. In this topic, you will learn more about quadratic functions. The diagram below shows the trajectory (path) of a soccer ball that was kicked in the air during a soccer match. The horizontal axis shows the time (in seconds) after the ball was kicked, and the vertical axis shows the height of the ball above the ground (in metres).



- How long did the ball take to reach the highest point above the ground?
- What was the maximum height that the ball reached above the ground?
- How long did the ball take to reach the ground again?
- What do you notice about the value $1\frac{1}{2}$ in terms of the numbers shown on the horizontal axis?

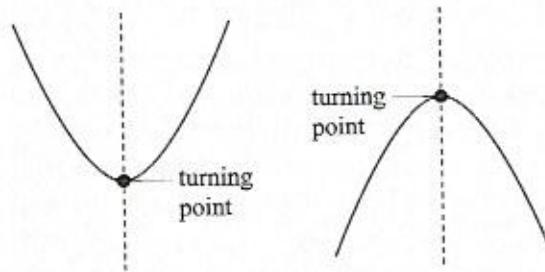
The quadratic function

A **quadratic function** is a function of the form $f(x) = ax^2 + bx + c$, where a , b and c are constants and $a \neq 0$. The graph of a quadratic function is called a **parabola**. A parabola has one of the shapes below, depending on the sign of a .



If a is positive, then the parabola has the shape of a smile (\cup). If a is negative, then the parabola has the shape of a frown (\cap).

Every parabola has a **turning point** and is symmetrical about the vertical line that passes through the turning point, as shown in the diagrams below.



This vertical line is called the **axis of symmetry** of the parabola.

The turning point of a parabola

The x -coordinate of the turning point of $y = ax^2 + bx + c$ is given by the formula $x = -\frac{b}{2a}$. This is also the equation of the axis of symmetry of the parabola. The y -coordinate of the turning point of $y = ax^2 + bx + c$ can be found by substituting the x -coordinate of the turning point into the equation of the parabola. Alternatively, we can use the formula $y = \frac{(4ac - b^2)}{4a}$ to find the y -coordinate of the turning point.

Note that the y -coordinate of the turning point is the minimum or maximum value of the function $y = ax^2 + bx + c$.

New words

quadratic function: a function of the form $f(x) = ax^2 + bx + c$, where a , b and c are constants and $a \neq 0$

parabola: the graph of a quadratic function

turning point: the point where a parabola turns; the minimum or maximum point of the parabola

axis of symmetry: the vertical line that passes through the turning point of a parabola

Worked example 1

Given the function $f(x) = 2x^2 + 4x - 5$.

- Will this function have a minimum or a maximum value?
- Determine the x -coordinate of the turning point.
- Determine the y -coordinate of the turning point.

Answers

- The coefficient of x^2 is positive, so the parabola has the shape of a smile.
This means that the parabola will have a minimum value.

- $a = 2$ and $b = 4$

$$\begin{aligned}\therefore x &= -\frac{b}{2a} \\ &= -\frac{4}{2(2)} \\ &= -1\end{aligned}$$

- We can find the y -coordinate of the turning point by substituting the value of -1 into the equation of the parabola, as follows:

$$\begin{aligned}y &= 2x^2 + 4x - 5 \\ &= 2(-1)^2 + 4(-1) - 5 \\ &= 2 - 4 - 5 \\ &= -7\end{aligned}$$

Alternatively, we can use the formula $y = \frac{(4ac - b^2)}{4a}$, as follows:

$$a = 2, b = 4 \text{ and } c = -5$$

$$\begin{aligned}\therefore y &= \frac{(4ac - b^2)}{4a} \\ &= \frac{(4(2)(-5) - (4)^2)}{4(2)} \\ &= \frac{(-40 - 16)}{8} \\ &= \frac{-56}{8} \\ &= -7\end{aligned}$$

Activity 1

- Given the function $f(x) = 6x^2 - x - 15$.
 - Will this function have a minimum or a maximum value?
 - Determine the x -coordinate of the turning point.
 - Determine the y -coordinate of the turning point.
- Given the function $f(x) = -x^2 - 2x + 24$.
 - Will this function have a minimum or a maximum value?
 - Determine the x -coordinate of the turning point.
 - Determine the y -coordinate of the turning point.
- Determine the coordinates of the turning point of each of the following.

a) $y = x^2 + x + 1$	b) $y = 3x^2 - 4x + 5$	c) $y = -2x^2 + 5x - 2$
d) $y = -4x^2 + 3x + 6$	e) $y = -10 + 2x - x^2$	f) $y = 1 - 2x + 4x^2$

The x - and y -intercepts of a parabola

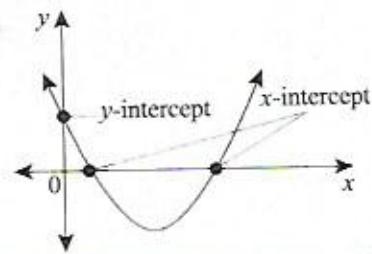
The parabola $y = ax^2 + bx + c$ intersects the y -axis at the point where $x = 0$ and the x -axis at the points where $y = 0$, as shown in the diagram alongside.

To find the y -intercept, set $x = 0$.

To find the x -intercepts, we use this formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This is called the **quadratic formula**. The x -intercepts of a parabola are also called the **roots** of the corresponding quadratic equation.



New words

quadratic formula: the formula used to find the x -intercepts of a parabola

roots (of a quadratic equation): the x -intercepts of the graph of the equation

Worked example 2

- 1 For a parabola with the equation $y = 2 - x - x^2$, find:
 - the y -intercept
 - the x -intercepts
 - the coordinates of the turning point.
- 2 Sketch the graph of $y = 2 - x - x^2$ and show the coordinates of the turning point, as well as all the intercepts with the axes.

Answers

- 1 First write the equation in the form $y = ax^2 + bx + c$, as follows: $y = -x^2 - x + 2$. So, $a = -1$, $b = -1$ and $c = 2$.

- To find the y -intercept, set $x = 0$.

$$\therefore y = 2$$

- To find the x -intercept, use the quadratic formula, where $a = -1$, $b = -1$ and $c = 2$.

$$\begin{aligned}x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\&= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(-1)(2)}}{2(-1)} \\&= \frac{1 \pm \sqrt{1+8}}{-2} \\&= \frac{1 \pm \sqrt{9}}{-2} \\&= \frac{1 \pm 3}{-2} \\&\therefore x = \frac{1+3}{-2} \text{ or } x = \frac{1-3}{-2} \\&\therefore x = \frac{4}{-2} \text{ or } x = \frac{-2}{-2} \\&\therefore x = -2 \text{ or } x = 1\end{aligned}$$

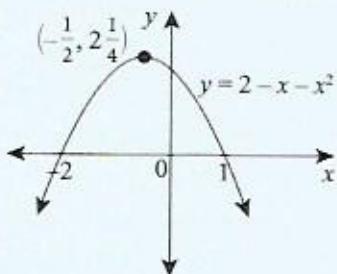
Worked example 2 (continued)

c) The x -coordinate of the turning point is: $x = -\frac{b}{2a} = -\frac{-1}{2(-1)} = -\frac{1}{2}$.

We can find the y -coordinate of the turning point by substituting the value of $-\frac{1}{2}$ into the equation of the parabola, or we can use the formula $y = \frac{(4ac - b^2)}{4a}$. Either way, we find that $y = 2\frac{1}{4}$.

The coordinates of the turning point are therefore $(-\frac{1}{2}, 2\frac{1}{4})$.

2



Activity 2

1 Find the roots of each of the following quadratic functions:

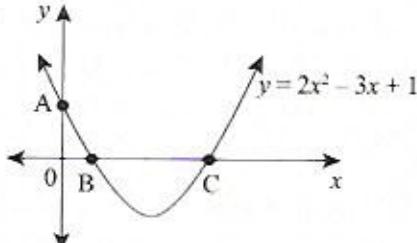
a) $y = x^2 + 5x + 6$ b) $y = 2x^2 + x - 3$ c) $y = 2x^2 + x - 6$
 d) $y = 3x^2 - 4x + 1$ e) $y = 12 - 4x - x^2$ f) $y = 10 - 3x - x^2$

2 Sketch the graph of each of the following. Show the coordinates of the turning point, as well as all the intercepts with the axes.

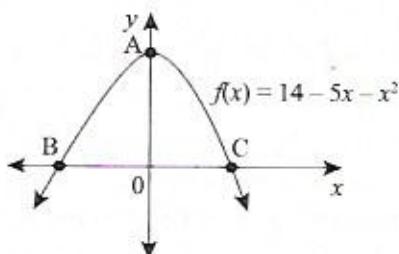
a) $y = x^2 + 5x - 6$ b) $y = 2 - x - 3x^2$
 c) $y = 2x^2 + 3x - 5$ d) $y = x^2 - 2x + 3$

3 Find the coordinates of the points A, B and C in each of the following.

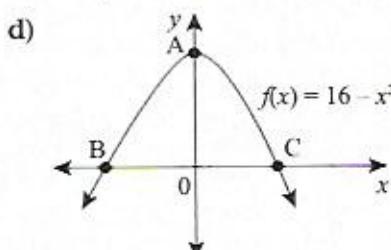
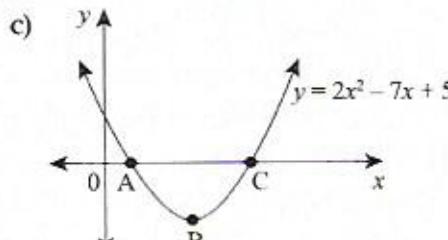
a)



b)



Activity 2 (continued)

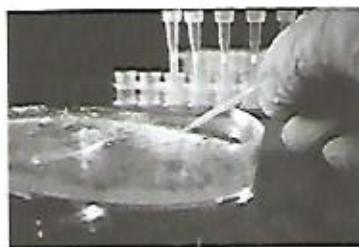


4 A parabola has the equation $y = x^2 - x - 6$.

- Find the y -intercept.
- Find the x -intercepts.
- Find the coordinates of the turning point.
- Sketch the parabola. Show the coordinates of the turning point, as well as all the intercepts with the axes.

5 The number of bacteria, n (in millions), in a certain colony after time t (in minutes) is defined by the quadratic function $n = t^2 - 7t + 14$. For this colony, find:

- the initial number of bacteria
- the number of bacteria present when $t = 2$
- the minimum number of bacteria.



A scientist working with a colony of bacteria in a Petri dish

6 A parabola has equation $f(x) = px^2 - qx - 30$.

- If $f(2) = -28$ and $f(3) = -24$, find the values of p and q .
- Sketch the graph of $f(x)$ and show the coordinates of the turning point, as well as all the intercepts with the axes.

Summary

Introduction to quadratic functions

- A quadratic function is a function of the form $f(x) = ax^2 + bx + c$, where a , b and c are constants and $a \neq 0$.
- The graph of a quadratic function is a parabola.
- If a is positive, then the parabola has the shape of a smile (\cup) and the parabola has a minimum point.
- If a is negative, then the parabola has the shape of a frown (\cap) and the parabola has a maximum point.
- Every parabola has a turning point and is symmetrical about the vertical line that passes through the turning point. This vertical line is the axis of symmetry of the parabola.
- The x -coordinate of the turning point of $y = ax^2 + bx + c$ is given by the formula $x = -\frac{b}{2a}$.
- The y -coordinate of the turning point of $y = ax^2 + bx + c$ can be found by substituting the x -coordinate of the turning point into the equation of the parabola. Alternatively, we can use the formula $y = \frac{(4ac - b^2)}{4a}$ to find the y -coordinate of the turning point.
- The y -coordinate of the turning point is the minimum or maximum value of the function $y = ax^2 + bx + c$.
- The parabola $y = ax^2 + bx + c$ intersects the y -axis at the point where $x = 0$ and the x -axis at the points where $y = 0$.
- To find the y -intercept, set $x = 0$.
- To find the x -intercepts, we use the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
- The x -intercepts of a parabola are also called the roots of the corresponding quadratic equation.

Revision exercises (remedial)

- 1 What is the graph of a quadratic function called? (1)
- 2 Given the function $f(x) = 3x^2 - 2x - 5$. Will this function have a minimum or a maximum value? Give a reason for your answer. (1)

Revision exercises

3 Find the maximum or minimum values of each of the following. For each one, say whether it is a maximum or a minimum value and give a reason.

- a) $y = 2x^2 + 4x - 1$ (3)
- b) $y = -5x^2 + 6$ (3)
- c) $y = 4x^2 + 2x - 5$ (3)
- d) $y = 9 - 3x - x^2$ (3)
- e) $y = x^2 + 5x + 6$ (3)
- f) $y = 3 + x - x^2$ (3)

4 Find the y -intercept and the x -intercepts of each of the following functions.

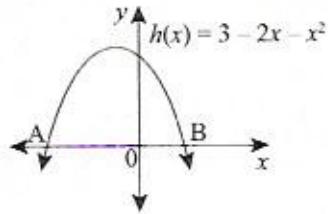
- a) $g(x) = 6 + x - 2x^2$ (3)
- b) $f(x) = 3x^2 + x$ (3)
- c) $f(x) = x^2 - 5x + 6$ (3)
- d) $f(x) = 12 - x - x^2$ (3)
- e) $h(x) = (x + 3)(x - 2)$ (3)

Total marks: 35

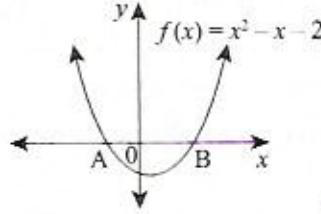
Assessment exercises

1 Find the coordinates of the points A and B in each of the following.

a)



b)



(2 x 3 = 6)

2 Given the function $g(x) = 2x^2 - x - 15$.

- a) Write down the y -intercept of $g(x)$. (1)
- b) Determine the x -intercepts of $g(x)$. (2)
- c) Determine the coordinates of the turning point of $g(x)$. (2)
- d) Sketch the graph of $g(x)$. Show the coordinates of the turning point, as well as all the intercepts with the axes. (5)

3 A parabola has equation $f(x) = px^2 - qx - 14$. If $f(-1) = 9$ and $f(4) = 14$, find the values of p and q . (4)

Total marks: 20

Quadratic equations

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Sub-topics	Specific Outcomes
Introduction to quadratic equations	<ul style="list-style-type: none"> Explain the meaning of the quadratic equation
Solutions of quadratic equations	<ul style="list-style-type: none"> Solve quadratic equations by the graphical method Solve quadratic equations using the factorisation method Solve quadratic equations using the completing the square method Solve quadratic equations using the quadratic formula method Apply quadratic equations to solve real-life problems

Starter activity

Work in small groups for this activity.

In Grade 10, you learned how to factorise quadratic expressions. Look at each of these examples and decide which solution (A, B, C or D) is correct.

1 $3x^2 + 9x - 15 =$	A. $3x(x + 3 - 5)$	B. $3(x^2 + 3x - 5)$
	C. $3(x^2 + 9x - 15)$	D. None of these
2 $x^2 - 16 =$	A. $(x - 4)(x - 4)$	B. $(x + 8)(x - 8)$
	C. $(x + 4)(x - 4)$	D. None of these
3 $x^2 - 2x + 3 =$	A. $(x - 3)(x + 1)$	B. $(x + 3)(x - 2)$
	C. $(x + 3)(x - 1)$	D. None of these
4 $x^2 - 5x - 6 =$	A. $(x - 3)(x - 2)$	B. $(x + 3)(x - 2)$
	C. $(x - 2)(x + 3)$	D. None of these
5 $4x^2 - 100 =$	A. $(2x - 5)(2x - 5)$	B. $4(x + 5)(x - 5)$
	C. $4(x + 10)(x - 10)$	D. None of these
6 $3x^2 - 3x - 36 =$	A. $(3x - 9)(x + 4)$	B. $3(x + 12)(x - 1)$
	C. $3(x - 4)(x + 3)$	D. None of these

Introduction

In Topic 5, you learnt about quadratic functions and how to draw the graph of a quadratic function. In this topic, the focus is on solving quadratic equations.

A **quadratic equation** is an equation of the form $ax^2 + bx + c = 0$, where a , b and c are constants and $a \neq 0$. x is the unknown quantity and solving a quadratic equation

involves finding the values of x which satisfy the equation (make it true). The values which satisfy the equation are known as the **roots** of the equation.

You also learned that the graph of any function of the form $y = ax^2 + bx + c$ where a , b and c are constants and $a \neq 0$, is a parabola.

When you found the x -intercepts of a parabola in Topic 5, you used the quadratic formula ($x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$). In this topic, you will learn other methods of solving a quadratic equation, including the graphical method, the factorisation method and the completing the square method. You will see how the quadratic formula is derived and you will also apply quadratic equations to solve real-life problems.

Activity 1

Work in pairs for this activity.

- 1 Factorise each of the following by taking out a common factor.
 - $5x^2 - 10x + 25$
 - $12x^2 + 18x - 6$
 - $21x^2 - 7x + 14$
- 2 Factorise each of the following, using the difference of squares.
 - $49x^2 - 144$
 - $169 - 4x^2$
 - $81x^2 - 1$
- 3 Factorise each of the following, by writing each one in the form $(x + \square)(x + \square)$.
 - $x^2 + 5x + 4$
 - $x^2 + 7x + 12$
 - $x^2 + 8x + 16$
- 4 Factorise each of the following, by writing each one in the form $(x - \square)(x - \square)$.
 - $x^2 - 9x + 18$
 - $x^2 - 10x + 25$
 - $x^2 - 5x + 6$
- 5 Factorise each of the following, by writing each one in the form $(x \pm \square)(x \pm \square)$.
 - $x^2 - 4x - 32$
 - $x^2 - x - 20$
 - $x^2 + 6x - 27$
 - $x^2 - 9x - 22$
 - $x^2 - 8x - 48$
 - $x^2 + 2x - 35$

New words

quadratic equation: an equation of the form $ax^2 + bx + c = 0$, where a , b and c are constants and $a \neq 0$

roots (of an equation): the values of x that satisfy an equation

Introduction

In this sub-topic, you will use the following methods of solving quadratic equations:

- the graphical method
- the factorisation method
- the completing the square method and
- the quadratic formula method.

The first method involves drawing graphs and the last three methods use algebraic manipulation.

Solve quadratic equations by the graphical method

If $y = f(x)$ is any function, the equation $f(x) = 0$ is satisfied by the values of x at the points where the graph of $f(x)$ crosses or touches the x -axis (where $y = 0$).

If we plot two or more functions on the same system of axes, we can also find interesting results from the points of intersection of the graphs.

Worked example 1

- 1 a) On the same system of axes, draw the graphs of $y = x^2 + x - 2$ and $y = -x - 2$ for $-3 \leq x \leq 2$.
 b) Use your graphs in Question a) to solve the equation $x^2 + x - 2 = 0$.
 c) The x -coordinates of the points of intersection of the graphs of $y = x^2 + x - 2$ and $y = -x - 2$ are the roots of a new equation.
 - Find this equation algebraically.
 - Use your graphs in Question a) to solve the equation in Question c)(i).
- 2 a) On the same system of axes, draw the graphs of $y = x^2 - 2x$ and $y = -x^2 + 4$.
 b) Use your graphs in Question a) to solve the equations:
 - $x^2 - 2x = 0$
 - $-x^2 + 4 = 0$
 c) The x -coordinates of the points of intersection of the graphs of $y = x^2 - 2x$ and $y = -x^2 + 4$ are the roots of a new equation.
 - Find this equation algebraically.
 - Use your graphs in Question a) to solve the equation in Question c)(i).

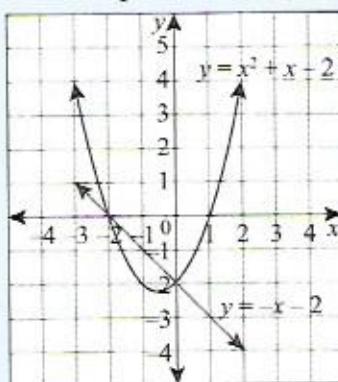
Answers

1 a) We first prepare a table of values of x and y for each function in the given range, as shown alongside.

x	-3	-2	-1	0	1	2
$y = x^2 + x - 2$	4	0	-2	-2	0	4
$y = -x - 2$	1	0	-1	-2	-3	-4

Worked example 1 (continued)

Next, we plot the two graphs, as shown below.



b) The solutions to the equation $x^2 + x - 2 = 0$ are given by the x -intercepts of the graph. The x -intercepts are -2 and 1 .
 So, $x = -2$ or $x = 1$.

c) (i) If we set $x^2 + x - 2 = -x - 2$, then we get: $x^2 + 2x = 0$.
 (ii) The roots of $x^2 + 2x = 0$ are the x -coordinates at the points of intersection of the two graphs.
 The graphs intersect where $x = -2$ and $x = 0$. So, $x = -2$ or $x = 0$.

b) (i) $x = 0$ or $x = 2$
 (ii) $x = -2$ or $x = 2$

c) (i) If we set $x^2 - 2x = -x^2 + 4$, then we get: $2x^2 - 2x - 4 = 0$, or $x^2 - x - 2 = 0$.
 (ii) The roots of $x^2 - x - 2 = 0$ are the x -coordinates at the points of intersection of the two graphs.
 The graphs intersect where $x = -1$ and $x = 2$.
 So, $x = -1$ or $x = 2$.

Activity 2

Activity 2 (continued)

d) The x -coordinates of the points of intersection of the graphs of $f(x)$ and $h(x)$ are the roots of a new equation.

- Find this equation algebraically.
- Use your graphs in Question a) to solve the equation in Question d)i).

Solve quadratic equations using the factorisation method

The rest of this topic deals with algebraic methods that we can use to solve quadratic equations. The first, and easiest method, is to use factorisation to solve a quadratic equation. To do this, we use the **zero-product rule**, which states: If $A \times B = 0$, then A must be zero or B must be zero. (Of course, both can be zero as well.)

New word

zero-product rule:
if $A \times B = 0$, then A must be zero or B must be zero

Worked example 2

Solve each of the following equations.

$$\begin{array}{lll} 1 \ x(x+2)=0 & 2 \ (x-3)(x-2)=0 & 3 \ x^2-36=0 \\ 4 \ x^2+7x+12=0 & 5 \ 4p^2-10p-6=0 & 6 \ m(m-3)=40 \end{array}$$

Answers

1 $x(x+2)=0$
 $\therefore x=0$ or $x+2=0$ Apply the zero-product rule.
 $\therefore x=0$ or $x=-2$

2 $(x-3)(x-2)=0$
 $\therefore x-3=0$ or $x-2=0$ Apply the zero-product rule.
 $\therefore x=3$ or $x=2$

3 $x^2-36=0$
 $\therefore (x+6)(x-6)=0$ Factorise the difference of squares.
 $\therefore x+6=0$ or $x-6=0$ Apply the zero-product rule.
 $\therefore x=-6$ or $x=6$

4 $x^2+7x+12=0$
 $\therefore (x+3)(x+4)=0$ Factorise the quadratic expression.
 $\therefore x+3=0$ or $x+4=0$ Apply the zero-product rule.
 $\therefore x=-3$ or $x=-4$

5 $2p^2-5p-3=0$
 $\therefore (p-3)(2p+1)=0$ Factorise the quadratic expression.
 $\therefore p-3=0$ or $2p+1=0$ Apply the zero-product rule.
 $\therefore p=3$ or $p=-\frac{1}{2}$

Note

We commonly use the variable x in quadratic equations with one variable, but we can use any other variable as well. In this case, we used p .

Worked example 2 (continued)

$$6 \ m(m-3) = 40$$

$$\therefore m^2 - 3m - 40 = 0$$

Rewrite the equation in the form

$$ax^2 + bx + c = 0.$$

$$\therefore (m-8)(m+5) = 0$$

Factorise the quadratic expression.

$$\therefore m-8 = 0 \text{ or } m+5 = 0$$

Apply the zero-product rule.

$$\therefore m = 8 \text{ or } m = -5$$

Activity 3

1 Solve the following factorised equations.

a) $x(x-4) = 0$

b) $x(2x+3) = 0$

c) $y(2-y) = 0$

d) $(x+2)(x-5) = 0$

e) $(x-2)(x-2) = 0$

f) $(2-3x)(3+x) = 0$

2 Solve the following quadratic equations by factorising.

a) $x^2 - 1 = 0$

b) $x^2 - 16 = 0$

c) $8x^2 - 18 = 0$

d) $1 - 25x^2 = 0$

e) $x^2 + 3x + 2 = 0$

f) $x^2 + 6x + 8 = 0$

g) $y^2 - 4y + 3 = 0$

h) $x^2 + x - 2 = 0$

i) $x^2 - x - 6 = 0$

j) $p^2 + p - 6 = 0$

k) $x^2 + 2x - 24 = 0$

l) $m^2 - m - 30 = 0$

m) $2p^2 + 5p + 2 = 0$

n) $2x^2 - 11x + 5 = 0$

3 Solve the following equations.

a) $x^2 - 3x = 28$

b) $(x-3)^2 = 4$

c) $6(x^2 - 1) = 5x$

d) $x(x+8) = 153$

e) $y^2 + 15 = 8y$

f) $9y^2 + 6 = 15y$

g) $2x^2 + x = 15$

h) $3(2n^2 + 1) = 11n$

i) $2 = x(x+1)$

j) $3(3-x) - 2x^2 = 0$

k) $6x(x+1) = 180$

l) $100 = 5x(x-1)$

Solve quadratic equations by completing the square

The method of completing the square of a quadratic equation is the method that was used to derive the quadratic formula. You will use the method of completing the square only when you are specifically asked to do so. If not, it is much quicker and simpler to use the quadratic formula instead.

Here are the steps to follow when using the completing the square method:

Step 1: Get the equation in the form $ax^2 + bx + c = 0$.

Step 2: Take the constant term across to the RHS (right-hand side) of the equation.

Step 3: Divide throughout by a to make the coefficient of x^2 equal to 1.

Step 4: Halve the coefficient of x , square the result and add the square to both sides of the equation.

Step 5: Factorise the LHS (left-hand side) of the equation. It will now be of the form $(x+k)^2$, where k is a constant.

Step 6: Take the square root on both sides of the equation. The LHS will become $x+k$. Remember to write a \pm in front of the square root on the RHS.

Step 7: Take the constant, k , across to the RHS.

Step 8: Calculate (or simplify) the roots, where one root takes the + and the other root takes the - in the \pm part of the solution.

Worked example 3

Solve each of the following equations by completing the square.

1 $x^2 - 2x - 15 = 0$

2 $2x^2 + 7x + 3 = 0$

3 $x^2 + 5x - 10 = 0$

Answers

1 $x^2 - 2x - 15 = 0$

$\therefore x^2 - 2x = 15$

$\therefore x^2 - 2x + (-1)^2 = 15 + (-1)^2$

$\therefore (x - 1)^2 = 16$

$\therefore x - 1 = \pm \sqrt{16}$

$\therefore x = 1 \pm 4$

$\therefore x = 1 + 4 \text{ or } x = 1 - 4$

$\therefore x = 5 \text{ or } x = -3$

2 $2x^2 + 7x + 3 = 0$

$\therefore 2x^2 + 7x = -3$

$\therefore x^2 + \frac{7}{2}x = -\frac{3}{2}$

$\therefore x^2 + \frac{7}{2}x + \left(\frac{7}{4}\right)^2 = -\frac{3}{2} + \left(\frac{7}{4}\right)^2$

$\therefore \left(x + \frac{7}{4}\right)^2 = -\frac{3}{2} + \frac{49}{16}$

$\therefore \left(x + \frac{7}{4}\right)^2 = \frac{25}{16}$

$\therefore x + \frac{7}{4} = \pm \sqrt{\frac{25}{16}}$

$\therefore x = -\frac{7}{4} \pm \frac{5}{4}$

$\therefore x = -\frac{7}{4} + \frac{5}{4} \text{ or } x = -\frac{7}{4} - \frac{5}{4}$

$\therefore x = -\frac{1}{2} \text{ or } x = -3$

This equation is in the correct form and the coefficient of x^2 is 1.

Take the constant term across to the RHS.

Halve the coefficient of x , square the result and add the square to both sides of the equation.

Factorise the LHS and simplify the RHS.

Take the square root on both sides of the equation.

Take the constant term across to the RHS.

Calculate the roots.

Note

If we factorise $x^2 - 2x - 15 = 0$ in the usual way, we get $(x - 5)(x + 3) = 0$, so $x = 5$ or $x = -3$. This confirms our answers.

This equation is in the correct form.

Take the constant term across to the RHS.

Divide throughout by 2 to make the coefficient of x^2 equal to 1.

Halve the coefficient of x , square the result and add the square to both sides of the equation.

Factorise the LHS and simplify the RHS.

Take the square root on both sides of the equation.

Take the constant term across to the RHS.

Calculate the roots.

Worked example 3 (continued)

$$3 \quad x^2 + 5x - 10 = 0$$

$$\therefore x^2 + 5x = 10$$

$$\therefore x^2 + 5x + \left(\frac{5}{2}\right)^2 = 10 + \left(\frac{5}{2}\right)^2$$

$$\therefore (x + \frac{5}{2})^2 = 10 + \frac{25}{4}$$

$$\therefore (x + \frac{5}{2})^2 = \frac{65}{4}$$

$$\therefore x + \frac{5}{2} = \pm \sqrt{\frac{65}{4}}$$

$$\therefore x = -\frac{5}{2} \pm \sqrt{\frac{65}{2}}$$

$$\therefore x = -\frac{5}{2} + \sqrt{\frac{65}{2}} \text{ or } x = -\frac{5}{2} - \sqrt{\frac{65}{2}}$$

$$\therefore x = 1.53 \text{ or } x = -6.53$$

This equation is in the correct form and the coefficient of x^2 is 1.

Take the constant term across to the RHS.

Halve the coefficient of x , square the result and add the square to both sides of the equation.

Factorise the LHS and simplify the RHS.

Take the square root on both sides of the equation.

Take the constant term across to the RHS.

Calculate the roots.

Give the answers correct to two decimal places.

Activity 4

1 Solve each of the following equations by completing the square. Use the factorising method to check your answers.

a) $x^2 + 7x + 10 = 0$

b) $x^2 - 3x - 10 = 0$

c) $x^2 + x - 2 = 0$

d) $x^2 - x - 12 = 0$

e) $2x^2 + x - 6 = 0$

f) $3x^2 - 4x - 15 = 0$

g) $2k^2 - k - 3 = 0$

h) $3p^2 + 2p - 5 = 0$

2 Solve each of the following equations by completing the square. Give your answers correct to two decimal places.

a) $x^2 + 5x - 12 = 0$

b) $x^2 - 6x - 1 = 0$

c) $x^2 + 2x - 6 = 0$

d) $2x^2 - x - 4 = 0$

e) $3x^2 + 2x - 3 = 0$

f) $3x^2 + 2x - 7 = 0$

g) $4t^2 + 8t - 1 = 0$

h) $p^2 + 6p + 1 = 0$

Solve quadratic equations using the quadratic formula method

We use the quadratic formula to solve quadratic equations where the factors cannot easily be found or where it is not possible to factorise the expression. You already know that, if $ax^2 + bx + c = 0$, then the roots of the equation are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This formula can be derived using the method of completing the square of a quadratic equation, as shown below.

$$ax^2 + bx + c = 0$$

This equation is in the correct form.

$$\therefore ax^2 + bx = -c$$

Take the constant term across to the RHS.

$$\therefore x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Divide throughout by a to make the coefficient of x^2 equal to 1.

$$\therefore x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

Halve the coefficient of x , square the result and add the square to both sides of the equation.

$$\therefore \left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

Factorise the LHS and simplify the RHS.

$$\therefore \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Take the square root on both sides of the equation.

$$\therefore x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

Take the constant term across to the RHS.

$$\therefore x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Simplify the RHS.

So, the formula for solving any equation of the form $ax^2 + bx + c = 0$ is:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Worked example 4

Use the quadratic formula to solve each of the following, correct to two decimal places.

$$1 \quad x^2 + 5x + 2 = 0$$

$$2 \quad 3x^2 - 5x + 1 = 0$$

$$3 \quad 2x^2 + 3x + 9 = 0$$

Answers

$$1 \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$2 \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substitute $a = 1$, $b = 5$ and $c = 2$ in the formula:

$$\therefore x = \frac{-5 \pm \sqrt{5^2 - 4(1)(2)}}{2(1)}$$

$$\therefore x = \frac{-5 \pm \sqrt{17}}{2}$$

$$\therefore x = \frac{-5 + 4.123}{2} \text{ or } x = \frac{-5 - 4.123}{2}$$

$$\therefore x = -0.44 \text{ or } x = -4.56$$

Substitute $a = 3$, $b = -5$ and $c = 1$ in the formula:

$$\therefore x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(1)}}{2(3)}$$

$$\therefore x = \frac{5 \pm \sqrt{13}}{6}$$

$$\therefore x = \frac{5 + 3.606}{6} \text{ or } x = \frac{5 - 3.606}{6}$$

$$\therefore x = 1.43 \text{ or } x = 0.23$$

Worked example 4 (continued)

$$3 \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substitute $a = 2$, $b = 3$ and $c = 9$ in the formula:

$$\therefore x = \frac{-3 \pm \sqrt{3^2 - 4(2)(9)}}{2(2)}$$

$$\therefore x = \frac{-3 \pm \sqrt{-63}}{4}$$

\therefore no solution, because it is not possible to calculate the square root of a negative number.

So this equation has no real roots.

Activity 5

1 Use the quadratic formula to solve each of the following equations.

Where necessary, give your answers correct to two decimal places.

a) $x^2 + x - 6 = 0$	b) $x^2 + x - 30 = 0$	c) $x^2 + 3x + 2 = 0$
d) $2x^2 + 6x + 3 = 0$	e) $3x^2 - 7x - 1 = 0$	f) $x^2 + x - 5 = 0$
g) $2x^2 - x - 14 = 0$	h) $5x^2 - 3x - 4 = 0$	i) $7 + 2x - 3x^2 = 0$
j) $2x^2 - 7x = 5$	k) $17x - 8 = 4x^2$	l) $x^2 + 5x + 1 = 0$
m) $2x^2 = 13(x - 1)$	n) $10 - x^2 = x$	

2 Use the quadratic formula to solve each of the following equations.

Where necessary, give your answers correct to two decimal places.

a) $3k^2 - 9k - 19 = 0$	b) $2p^2 + 11p + 5 = 0$	c) $2m^2 + m - 5 = 0$
d) $a^2 - a - 3 = 0$	e) $2y^2 + 4y - 5 = 0$	f) $4v^2 - 2v - 1 = 0$
g) $f^2 + f - 3 = 0$	h) $2b^2 + 8b + 5 = 0$	i) $2h^2 - 13h + 12 = 0$
j) $3r^2 + 6r + 1 = 0$	k) $12 + 15m - 2m^2 = 0$	l) $2n(n - 5) = 3(n^2 + 2n + 4)$

Apply quadratic equations to solve real-life problems

Worked example 5

1 If the sum of two numbers is 17 and their product is 72, find the numbers.

2 The length of a rectangle is 2 cm greater than its breadth. If the area of the rectangle is 48 cm², find its dimensions.

Answers

1 Let the numbers be x and $(17 - x)$.

$$\therefore x(17 - x) = 72$$

$$\therefore 17x - x^2 = 72$$

$$\therefore x^2 - 17x + 72 = 0$$

$$\therefore (x - 8)(x - 9) = 0$$

$$\therefore x = 8 \text{ or } x = 9$$

If $x = 8$, $y = 17 - 8 = 9$.

If $x = 9$, $y = 17 - 9 = 8$.

So, the numbers are 8 and 9.

Worked example 5 (continued)

2 If the length of the rectangle is x , then the breadth is $x - 2$.

Area = length \times breadth

$$\therefore 48 = x(x - 2)$$

$$\therefore x^2 - 2x - 48 = 0$$

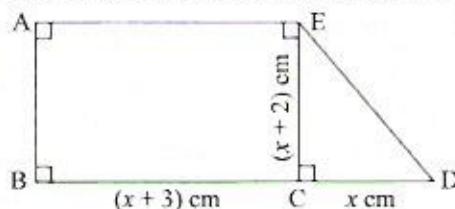
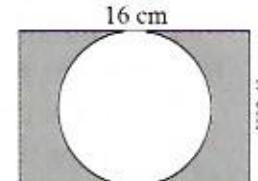
$$\therefore (x - 8)(x + 6) = 0$$

$$\therefore x = 8 \text{ or } x = -6$$

The length of the rectangle cannot be negative, so the length of the rectangle is 8 cm and the breadth is $8 - 2 = 6$ cm.

Activity 6

- 1 The difference between two numbers is 8 and their product is 20. Find the numbers.
- 2 The difference between two numbers is 5 and the sum of their squares is 73. Find the numbers.
- 3 The sum of two numbers is 16 and the difference between their squares is 32. Find the numbers.
- 4 The product of two consecutive numbers is 306. Find the numbers.
- 5 In a right-angled triangle, the lengths of the two non-hypotenuse sides differ by 2 cm. If the area of the triangle is 84 cm², find the lengths of these sides.
- 6 The product of two consecutive even numbers is 168. Form an equation and solve it to find the two numbers.
- 7 The length of a rectangle is 7 cm greater than its breadth. If the area of the rectangle is 60 cm², find its dimensions.
- 8 The perimeter of a rectangle is 26 cm and the area is 42 cm². Find the dimensions of the rectangle.
- 9 The diagram alongside shows a rectangular piece of cardboard of length 16 cm and breadth x cm from which a circle of diameter x cm has been removed. If the area of the shaded region in the diagram is 70 cm², find the possible values of x . Use $\pi = \frac{22}{7}$.
- 10 ABDE is a trapezium with an area of 54 cm², as shown in the diagram below.



- a) Form an equation in x and show that it reduces to $x^2 + 4x - 32 = 0$.
- b) Solve this equation.
- c) Find the lengths of AB and BD.

Activity 6

11 The sum of the ages of a group of boys is 150 years. When two boys aged 23 leave the group, the average age drops by two years. Find all possible values of the average age of the remaining boys.

12 A shelf that is 90 cm long is full of identical books, each x cm thick.

- Write down an expression in terms of x for the number of books on the shelf.
- If the books are removed and replaced by other books 1 cm less in thickness, write down another expression for the number of books on the shelf, given that it is full.
- If the shelf contains 15 more books in Question b) than in Question a), form an equation in x and show that it reduces to $x^2 - x - 6 = 0$.
- Solve the above equation.
- How many books were on the shelf originally?

13 x girls share 20 oranges equally among themselves.

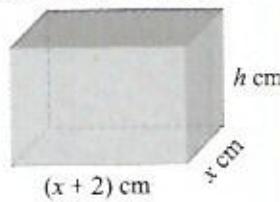
- Write down an expression in x for the number of oranges that each girl receives.
- If $(x - 5)$ girls share the same number of oranges, write down an expression in x for the number of oranges that each girl receives.
- If each girl in Question b) received 2 more oranges than in Question a), write down an equation in x and show that it reduces to $x^2 - 5x - 50 = 0$.
- Solve the above equation for x .
- How many oranges did each girl receive originally?

14 Nosiku travelled from Aibelilwe to Aikelile, 60 km apart, in two stages.

- For the first 30 km, she cycled at an average speed of y km/h. Write down an expression in y for the time she took to complete the first stage.
- For the last 30 km, Nosiku jogged at an average speed of 4 km/h less than her average cycling speed. Write down an expression in y for the time she took to jog the second stage.
- If Lindiwe took two hours longer to complete the second stage than she took to complete the first stage, find the average speed at which she cycled.

15 A closed box measures $(x + 2)$ cm by x cm by h cm, as shown in the diagram alongside.

- Given that the total surface area of the box is 52 cm^2 , show that $h = \frac{26 - 2x - x^2}{2x + 2}$.
- Given that $h = 3$, show that the above equation reduces to $x^2 + 8x - 20 = 0$.
- Solve the equation $x^2 + 8x - 20 = 0$.



Summary

Introduction to quadratic equations

- A quadratic equation is an equation of the form $ax^2 + bx + c = 0$, where a , b and c are constants and $a \neq 0$.
- The values which satisfy the equation are known as the roots of the equation.

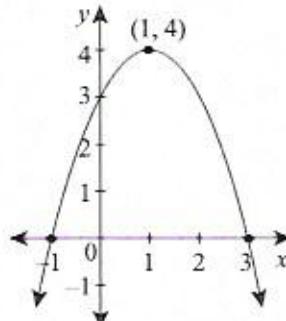
Solutions of quadratic equations

- The graph of any function of the form $y = ax^2 + bx + c$ where a , b and c are constants and $a \neq 0$, is a parabola.
- If $y = f(x)$ is any function, the equation $f(x) = 0$ is satisfied by the values of x at the points where the graph of $f(x)$ crosses or touches the x -axis (where $y = 0$).
- When we use factorisation to solve a quadratic equation, we use the zero-product rule, which states: If $A \times B = 0$, then A must be zero or B must be zero.
- The method of completing the square of a quadratic equation is the method that was used to derive the quadratic formula.
- You will only use the method of completing the square when you are specifically asked to do so.
- We use the quadratic formula to solve quadratic equations where the factors cannot easily be found or where it is not possible to factorise the expression.
- The quadratic formula states: If $ax^2 + bx + c = 0$, then the roots of the equation are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Revision exercises (remedial)

1 The graph of the function $y = -x^2 + 2x + 3$ is shown alongside. Use the graph to answer the following questions.



- For what values of x is $-x^2 + 2x + 3 = 0$? (2)
- What is the value of $-x^2 + 2x + 3$ if $x = 0$? (1)
- What is the value of x at the turning point of this function? (1)
- Does this function have a maximum or minimum value? (1)
- What is this maximum or minimum value? (1)

Revision exercises

2 Solve the following factorised equations.

a) $x(x + 1) = 0$ b) $2x(x - 3) = 0$
 c) $(x + 5)(x - 8) = 0$ d) $(3x + 1)(2x - 7) = 0$ $(4 \times 2 = 8)$

3 Solve the following equations by factorising.

a) $5x^2 - x = 0$ b) $9 - y^2 = 0$
 c) $100x^2 = 16$ d) $(4 - x)^2 = 9$
 e) $4(2 - m)^2 - 36 = 0$ f) $x^2 + 6x + 5 = 0$
 g) $x^2 + 4x - 12 = 0$ h) $3x^2 + x - 4 = 0$
 i) $12 - x - x^2 = 0$ j) $3 + x - 2x^2 = 0$ $(10 \times 2 = 20)$

4 Solve each of the following equations by completing the square.

a) $x^2 - 4x - 12 = 0$ b) $x^2 - 2x - 4 = 0$ $(2 \times 4 = 8)$

5 Use the quadratic formula to solve each of the following equations.

Where necessary, give your answers correct to two decimal places.

a) $x^2 + 6x + 2 = 0$ b) $x^2 + 7x + 4 = 0$
 c) $3 + 2x - 2x^2 = 0$ d) $2x^2 + 8x - 1 = 0$
 e) $5x^2 - x = 6$ f) $3x^2 + 3x - 5 = 0$ $(6 \times 3 = 18)$

Total marks: 60

Assessment exercises

1 Use any method to solve each of the following equations. Where necessary, give your answers correct to two decimal places.

a) $m^2 + 7m = 0$ b) $6 + x - 2x^2 = 0$
 c) $3x^2 = 4 + x$ d) $x^2 - 3x - 1 = 0$
 e) $3x^2 + 5x - 8 = 0$ f) $(x + 2)^2 - 4 = 0$
 g) $5m^2 - m - 6 = 0$ h) $3x^2 - 6x + 2 = 0$
 i) $2x^2 - 3x - 5 = 0$ j) $5 + 6x - 2x^2 = 0$ $(10 \times 2 = 20)$

2 Solve each of the following equations by completing the square.

a) $3 + x - 2x^2 = 0$ b) $3x^2 + 5x - 8 = 0$ $(2 \times 4 = 8)$

3 Given that $x = t + 1$ and $y = t^2 + t$.

a) Show that $4x^2 - 3y = t^2 + 5t + 4$. (2)
 b) Hence, find the values of t for which $8x^2 - 6y = 0$. (4)

4 The length of a rectangle is $(x + 10)$ cm and its area is $(2x^2 + 14x - 60)$ cm².

a) Express the breadth of the rectangle in terms of x . (3)
 b) For what values of x will the rectangle exist?
 (Hint: In other words, for what values of x will the rectangle have positive length and a positive breadth?)

Total marks: 40



Sub-topics	Specific Outcomes
Introduction to variation	<ul style="list-style-type: none"> Describe variation
Direct and inverse variation	<ul style="list-style-type: none"> Distinguish between direct and inverse variation
Joint and partial variation	<ul style="list-style-type: none"> Distinguish between joint and partial variation
Graphs	<ul style="list-style-type: none"> Draw and interpret graphs of variation
Applications	<ul style="list-style-type: none"> Solve problems involving variation

Starter activity

Work in small groups for this activity.

1 When describing an electrical circuit, we use these variables: R , I and V .

R refers to resistance, I refers to current and V refers to voltage.

Given that $R = \frac{V}{I}$, complete each of the following:

a) $V = \square$ b) $I = \square$

2 A current of 2 amperes flows in a conductor whose resistance to electrical current is 3 ohms.

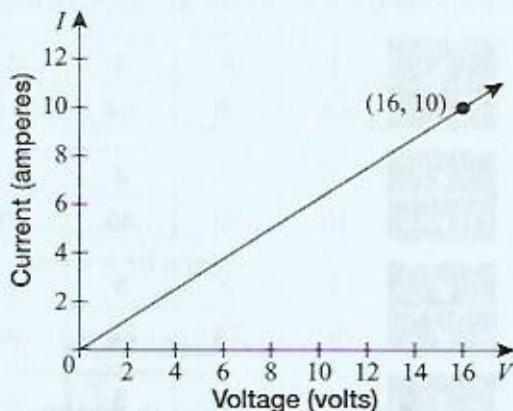
a) Calculate the potential difference (voltage) across the conductor.

b) What would the current be if the potential difference across the conductor is 8 volts?

3 Look at the voltage-current graph alongside.

a) Describe the relationship between I and V .

b) Given that $R = \frac{V}{I}$, what is the value of R in this case? Make sure that you write the units of measurement correctly in your answer.



In everyday language, a variation is a change to something. The term "variation" is used in different fields in different ways. For example, in biology, a variation can mean a difference within a species. In physics, the magnetic variation is the difference between magnetic north and true north and in music, a variation on a theme is a repetition of the basic theme, but with an interesting twist to the theme.

In mathematics, we use the term "variation" to describe specific relationships between two or more variables. In this topic you will learn about the following kinds of variation:

- direct variation
- inverse variation
- joint variation and
- partial variation.

Activity 1

Work in pairs for this activity.

For each of the following pairs of variables, find a formula to calculate y in terms of x . Then say whether the y -values are increasing or decreasing as the x -values are increasing.

Example:

x	1	2	3	4	5	6
y	1	4	9	16	25	36

The formula is $y = x^2$. As the x -values increase, the y -values increase as well.

1	x	1	2	3	4	5	6
	y	-3	-6	-9	-12	-15	-18

2	x	1	2	3	4	5	6
	y	10	20	30	40	50	60

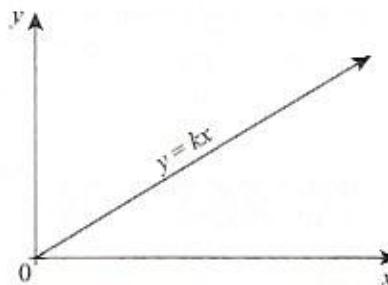
3	x	1	2	3	4	5	6
	y	6	16	26	36	46	56

4	x	1	2	3	4	5	6
	y	30	15	10	7.5	6	5

1	x	1	2	3	4	5	6
	y	-3	-6	-9	-12	-15	-18
2	x	1	2	3	4	5	6
	y	10	20	30	40	50	60
3	x	1	2	3	4	5	6
	y	6	16	26	36	46	56
4	x	1	2	3	4	5	6
	y	30	15	10	7.5	6	5

Direct variation

In a **direct variation**, as one variable increases, the other variable increases as well. Look at the graph alongside. As the value of x increases, the value of y increases as well. We say that y varies directly with x . We write this as: $y \propto x$, where the symbol \propto means "varies directly". Written as an equation, this becomes $y = kx$, where $k > 0$. k is called the **constant of variation**.

**New words**

direct variation: a relationship between two variables, such that as one variable increases, the other increases

constant of variation: the constant, k , in a variation equation

Worked example 1

Given that y varies directly with x , and $y = 8$ when $x = 6$.

- Find k (the constant of variation).
- Find y when $x = 3$.
- Find x when $y = 24$.

Answers

- $y = kx$
 $\therefore k = \frac{y}{x} = \frac{8}{6} = \frac{4}{3}$
- $y = \frac{4}{3}x = \frac{4}{3}(3) = 4$
- $x = \frac{3}{4}y = \frac{3}{4}(24) = 18$

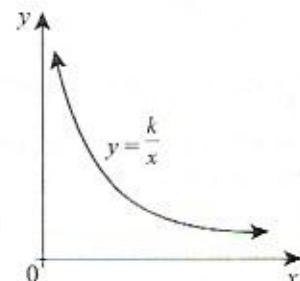
Activity 2

- Given that y varies directly with x , and $x = 10$ when $y = 4$.
 - Find k (the constant of variation).
 - Determine the equation for this situation in the form $y = kx$.
 - Find y when $x = 6$.
- If P varies directly with Q , and the constant of variation = $\frac{2}{5}$, what is the value of P when $Q = 10$?

Inverse variation

In an **inverse variation**, as one variable increases, the other variable decreases. Look at the graph alongside.

As the value of x increases, the value of y decreases. We say that y varies inversely with x . We write this as: $y \propto \frac{1}{x}$. Written as an equation, this becomes $y = k(\frac{1}{x}) = \frac{k}{x}$, where $k > 0$ and k is the constant of variation.



New word

inverse variation: a relationship between two variables, such that as one variable increases, the other decreases

Worked example 2

Given that y varies inversely with x , and $y = 4$ when $x = 7$. Find:

- the constant of variation
- the equation for this situation in the form $y = \frac{k}{x}$
- the value of y , if $x = 2$.

Answers

$$\begin{aligned} \text{a) } y &\propto \frac{1}{x} & \text{b) } y &= \frac{k}{x} = \frac{28}{x} & \text{c) } y &= \frac{28}{x} = \frac{28}{2} \\ \therefore y &= \frac{k}{x} & & & \therefore y &= 14 \\ \therefore k &= xy = 7 \times 4 = 28 & & & & \end{aligned}$$

Activity 3

- Given that y varies inversely with x , and $y = 3$ when $x = 8$. Find:
 - the constant of variation
 - the equation for this situation in the form $y = \frac{k}{x}$
 - the value of y , if $x = 6$.
- Given that y varies inversely with x , and $y = 12$ when $x = 12$. Find:
 - the constant of variation
 - the equation for this situation in the form $y = \frac{k}{x}$
 - the value of x , if $y = 72$.
- Given that y varies inversely with x , and $y = 150$ when $x = 30$. Find:
 - the constant of variation
 - the equation for this situation in the form $y = \frac{k}{x}$
 - the value of y , if $x = 150$
 - the value of x , if $y = x$ and $x > 0$.

SUB-TOPIC 3

Joint and partial variation

Joint variation

A joint variation is a variation where a quantity varies directly with a product of two or more other quantities. For instance, if y varies directly with x and also with z , we say that y varies jointly with x and z . This is written as $y \propto xz$. Written as an equation, this becomes $y = kxz$, where $k > 0$ and k is the constant of variation.

New word

joint variation: a variation where a quantity varies directly with a product of two or more other quantities

Worked example 3

P varies jointly with Q and the square of R , and P is 3 when $Q = 2$ and $R = 3$.

- Calculate the constant of variation.
- Determine the equation that relates the three variables P , Q and R .
- Calculate the value of R when $P = 4$ and $Q = 1$.

Answers

a) $P = kQR^2$

$$\therefore k = \frac{P}{QR^2} = \frac{3}{(2)(3^2)} = \frac{3}{18} = \frac{1}{6}$$

b) $P = kQR^2 = \frac{1}{6}QR^2$

c) $P = \frac{1}{6}QR^2$

$$\therefore 4 = \frac{1}{6} \times 1 \times R^2$$

$$\therefore R^2 = 24$$

$$\therefore R = \pm \sqrt{24}$$

Activity 4

- y varies jointly with x and z . y is 12 when $x = 3$ and $z = 2$.
 - Find the constant of variation.
 - Find the equation that relates the three variables x , y and z .
 - Find the value of y when $x = z = 5$.
 - Find the value of z when $y = 40$ and $x = \frac{1}{2}$.
- M varies directly with t and inversely with s . $M = 24$ when $t = 3$ and $s = 2$.
 - Find the constant of variation.
 - Find the equation that relates the three variables M , t and s .
 - Find the value of M when $t = 5$ and $s = 8$.

Partial variation

If y varies partially with x , then the relationship between y and x can be expressed as $y = mx + b$, where b is the fixed part of the **partial variation** and m is the variable part of the partial variation.

The graph of a partial variation is a straight line that does not pass through the origin.

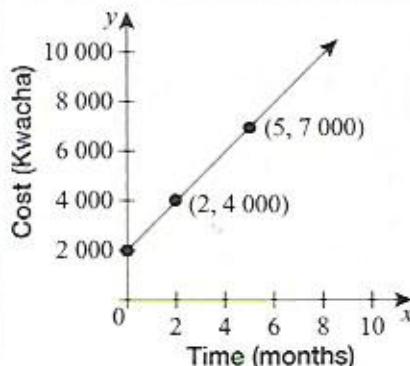
New word

partial variation: a variation between two variables, such that the graph of this variation is a straight line that does not pass through the origin

Worked example 5

The cost of renting a room is shown in the graph alongside.

- Determine the equation for this situation in the form $y = mx + b$.
- What will it cost to rent the room for 20 months?



Answers

a) $y = mx + b$

$b = 2\ 000$ (the y -intercept)

$$\begin{aligned} \text{From Topic 3, we know that } m &= \frac{(y_2 - y_1)}{(x_2 - x_1)} \\ &= \frac{(7\ 000 - 4\ 000)}{(5 - 2)} \\ &= 1\ 000 \end{aligned}$$

$$\therefore y = 1\ 000x + 2\ 000$$

b) If $x = 20$, then $y = (1\ 000 \times 20) + 2\ 000 = 22\ 000$.

So, it will cost K22 000 to rent the room for 20 months.

Activity 5

- Refer back to the previous example. If the Chona family paid K30 000 to rent the room, for how long did they rent it?
- The equation of a partial variation is $y = 3x + b$. $y = 23$ when $x = 3.5$. Determine the value of b .
- The following points lie on the graph of a partial variation: $(0, 6)$ and $(1, 10)$.
 - Determine the equation of the partial variation.
 - If the point $(13, y)$ lies on the same graph, calculate the value of y .
- Explain why the equation $y = \frac{1}{2}x$ is not the equation of a partial variation.
- The equation of a partial variation is $y = mx + b$. $y = 4$ when $x = 1$ and $y = 6$ when $x = 5$. Determine the values of m and b .

In this sub-topic, you will work with the graphs of different variations. Note the following points about the graphs of variations:

- The equation of a direct variation is $y = kx$. The graph of a direct variation is a straight line that passes through the origin. The value of k is equal to the gradient of the graph, m .
- The equation of a partially direct variation is $y = kx + b$. The graph of a direct variation is a straight line that does not pass through the origin. The value of k is equal to the gradient of the graph, m and the value of b is the y -intercept of the graph.
- The graph of an inverse variation is a curve that does not touch either of the axes.

Activity 6

1 The graph alongside shows the relationship between the voltage, V , and the current, I , of an electrical circuit.

- What kind of variation does the graph represent?
- Find k (the constant of variation).
- Determine the equation for this situation.
- Calculate the value of I if $V = 22$ volts.

2 $P(3, 12)$ is a point on a graph with $x = 3$ and $y = 12$. Determine the equation of the graph, if:

- y varies directly with x
- y varies inversely with x .

3 Given that y varies inversely with x and that the constant of variation is 36.

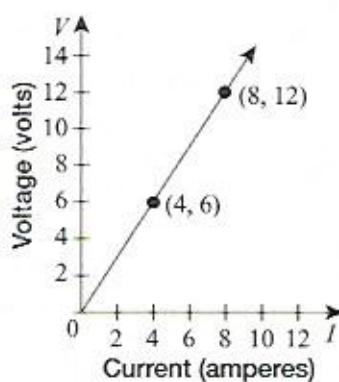
- Determine the equation for this situation.
- Use your equation to complete the table below.

x	1	2	3	4	6	9	12	18	36
y									

c) Sketch the graph of this variation.

4 Given that y varies partially with x and that the following points lie on the graph: $P(2, 8)$ and $Q(6, 20)$.

- Draw a neat sketch graph of this situation.
- Use your graph to find the equation of this partial variation.
- Use algebraic methods to test your answer to b).



In this sub-topic, you will apply your knowledge of variations as you solve a variety of problems set in real-life contexts.

Activity 7

- The voltage, V , varies directly with the current, I , and $V = 6$ volts when $I = 4$ amperes.
 - Find k (the constant of variation).
 - Write an equation that represents the relationship between V and I .
 - Find V when $I = 7$ amperes.
 - Find I when $V = 12$ volts.
- The number of hours, h , that it takes for ice to melt, varies inversely with the temperature, t . If it takes 3 hours for a piece of ice to melt at 45°C , find:
 - the constant of variation
 - the equation for this situation in the form $h = \frac{k}{t}$
 - the value of t , if $h = 4$ hours.
- On a certain map, 50 km is represented by 3 cm. If two towns are 24.5 cm apart on the map, what is the actual distance between the two towns?
- Mr Chirwa earns a monthly wage of K5 000 and 12% commission on the Kwacha value of his monthly clothing sales. Let y represent his total monthly earnings and x represent the total amount of his monthly clothing sales.
 - Determine the equation for this situation in the form $y = mx + b$.
 - How much will Mr Chirwa earn in a month during which he sells clothes worth K20 000?
 - What was the value of clothing that he sold in one month if his earnings for the month were K6 000?
- The force, F , needed to break a plank varies inversely with the length, l , of the plank. If it takes a force of 25 N to break a plank of 10 metres long, calculate:
 - the constant of variation
 - the value of l when $F = 10$ N.
- The volume, V , of a cone varies directly with its perpendicular height, h , as well as with the square of its radius, r . If $r = 10$ cm and $h = 30$ cm, then $V = \frac{22000}{7}$ cm³.
 - Find k (the constant of variation). Write your answer as an improper fraction.
 - Determine the equation for this situation in the form $V = \dots$.
 - If $V = 308$ mm³ and $r = 7$ mm, find the value of h .
 - If $V = \frac{44}{7}$ m² and $h = 1.5$ m, find the value of r .



Summary

Introduction to variation

- In everyday language, a variation is a change to something.
- In mathematics, we use the term “variation” to describe specific relationships between two or more variables.

Direct and inverse variation

- In a direct variation, as one variable increases, the other variable increases as well.
- If y varies directly with x , then $y \propto x$, and $y = kx$, where k is the constant of variation.
- In an inverse variation, as one variable increases, the other variable decreases.
- If y varies inversely with x , then $y \propto \frac{1}{x}$ and $y = \frac{k}{x}$, where k is the constant of variation.

Joint and partial variation

- A joint variation is a variation where a quantity varies directly with a product of two or more other quantities.
- If y varies jointly with x and z , then $y \propto kxz$ and $y = kxz$, where k is the constant of variation.
- If y varies partially with x , $y = mx + b$, where b is the fixed part of the partial variation and m is the variable part of the partial variation.

Graphs

- The equation of a direct variation is $y = kx$. The graph of a direct variation is a straight line that passes through the origin. The value of k is equal to the gradient of the graph, m .
- The equation of a partially direct variation is $y = kx + b$. The graph of a direct variation is a straight line that does not pass through the origin. The value of k is equal to the gradient of the graph, m and the value of b is the y -intercept of the graph.
- The graph of an inverse variation is a curve that does not touch either of the axes.

Applications

- You can apply your knowledge of variations to solving a variety of problems set in real-life contexts.

Revision exercises (remedial)

1 The equation for a certain variation is $y = kx$. When $x = 5$, $y = 20$.

- What kind of variation is this?
- Determine the value of k .
- Find the value of y if $x = 10$.
- Find the value of x if $y = 10$.

$(4 \times 1 = 4)$

Revision exercises

2 From the list in brackets, write down the name of each kind of variation next to its equation below (partial/inverse/joint/direct).

- $y = kxz$
- $y = mx + b$
- $y = kx$
- $y = \frac{k}{x}$

3 $P(2, 3)$ is a point on a graph of a variation. Find the equation of this graph in the form $y = \dots$, if the variation is:

- a direct variation
- an inverse variation.

4 The following points lie on the graph of a partial variation: $(0, 2)$ and $(4, 12)$.

- Determine the equation of the partial variation.
- If the point $(2.5, y)$ lies on the graph, calculate the value of y .

5 Speed (S), distance (D) and time (T) are related by the formula $S = \frac{D}{T}$.

- If S is constant, what kind of variation exists between D and T?
- If D is constant, what kind of variation exists between S and T?

Total marks: 20

Assessment exercises

1 Given that y varies inversely with x and that the constant of variation is 12.

- Determine the equation for this situation.
- Sketch the graph of this variation, for values of $0 < x \leq 12$.
- Explain why this equation is undefined for $x = 0$.

2 y varies jointly with x and z . y is 42 when $x = 2$ and $z = 7$.

- Find the equation that relates the three variables x , y and z .
- Find the value of z when $y = 18$ and $x = 1$.
- Find the value of y when $z = 18$ and $x = 1$.

3 Kepler's third law of planetary motion says that the square of the time that it takes for a planet to make one full revolution around the sun is directly proportional to the cube of the average distance of the planet from the sun. If the distance from Mars to the sun is 1.5 times the distance from the Earth to the sun, how long, in terms of Earth years, will it take Mars to make one full revolution around the sun?

Total marks: 20

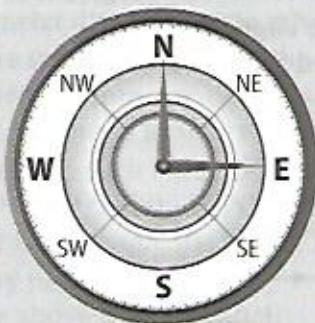
Sub-topics	Specific Outcomes
Properties of a circle	<ul style="list-style-type: none">Analyse the parts of a circle
Angle properties	<ul style="list-style-type: none">Solve problems using angle properties of a circleSolve problems involving tangent properties

Starter activity

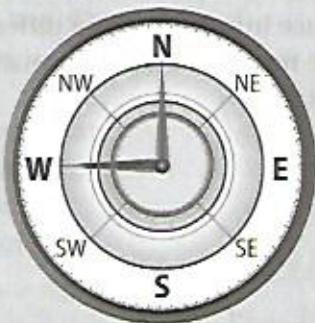
Work in pairs for this activity.

1 Each compass below indicates a specific direction. Find the size of the angle, clockwise from North, that has been formed in each case.

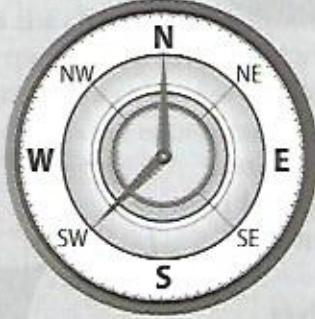
a)



b)



c)



2 Draw a compass to show the direction that would correspond with each of the following angles.

- a) 45°
- b) 180°
- c) 135°

Introduction

You are already familiar with the following parts of a circle: the centre, the diameter, the radius and the circumference. In this sub-topic, you will revise these terms and you will also learn about other terms that we use to describe the parts of a circle.

Analyse the parts of a circle

A chord is a straight line drawn across a circle that joins any two points on its circumference.

A chord that passes through the centre of a circle is called a **diameter**.

A line drawn from the centre of a circle to its circumference is called a **radius**. A diameter consists of two **radii**. See Figure 8.1.

A chord which is not a diameter divides the circumference into two **arcs** of different sizes. The larger arc is called the **major arc** and the smaller arc is called the **minor arc**. See Figure 8.2.

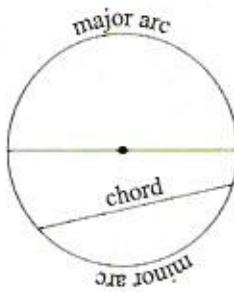


Figure 8.1

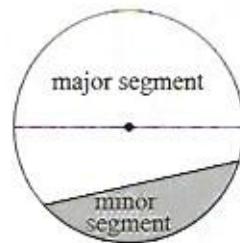


Figure 8.2

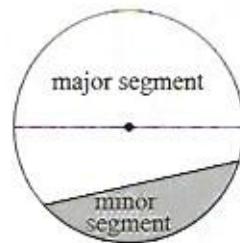


Figure 8.3

A chord divides the circle into two parts of different sizes. These portions are called **segments**. The larger segment is called the **major segment** and the smaller segment is called the **minor segment**. See Figure 8.3.

A diameter divides the circle into two equal halves. These are called **semi-circles**.

A sector of a circle is an area bounded by two radii and an arc. The two radii divide the circle into two **sectors**. The larger sector is called the **major sector** and the smaller sector is called the **minor sector**. See Figure 8.4.

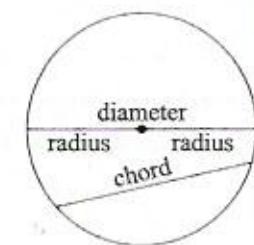


Figure 8.1

New words (continued)

chord: a straight line drawn across a circle that joins any two points on its circumference
diameter: a chord that passes through the centre of a circle
radius: a line drawn from the centre of a circle to its circumference
radii: the plural of radius
arc: a piece of the circumference of a circle
major arc: the larger arc formed by a chord that is not a diameter
minor arc: the smaller arc formed by a chord that is not a diameter
segment: a part of a circle that is enclosed by a chord and an arc
major segment: the larger segment formed by a chord that is not a diameter
minor segment: the smaller segment formed by a chord that is not a diameter
semi-circle: half a circle
sector: a part of a circle that is enclosed by two radii and an arc
major sector: the larger sector formed by two radii that are not part of the same diameter
minor sector: the smaller sector formed by two radii that are not part of the same diameter

The bilateral symmetry of a circle

Any diameter divides a circle into two equal halves. These halves are symmetrical about the diameter. This is called the **bilateral symmetry** of a circle.

Figure 8.5 shows a circle with centre O, chord PQ and diameter XY, such that XY is perpendicular to PQ. Since $OP = OQ$ (radii), $\triangle OPQ$ is an isosceles triangle. Since XY is perpendicular to PQ, XY is an axis of symmetry of the circle.

From the above it follows that:

- a diameter (or radius) drawn perpendicular to a chord bisects the chord. See Figure 8.6.
- the perpendicular bisector of a chord passes through the centre of a circle. See Figure 8.6.
- equal chords are equidistant from the centre of the circle. See Figure 8.7.

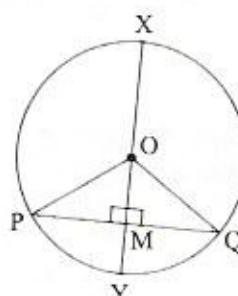


Figure 8.5

New word

bilateral symmetry:
 line symmetry;
 symmetry around
 a line

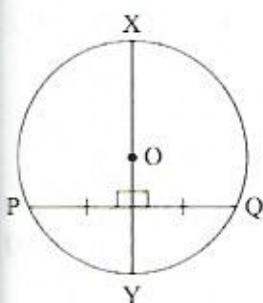


Figure 8.6

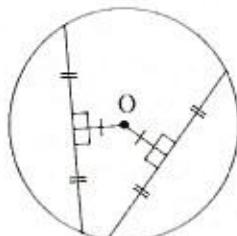


Figure 8.7

Worked example 1

In the diagram alongside, O is the centre of the circle, $OP = 10 \text{ cm}$ and $PQ = 16 \text{ cm}$. Calculate, with reasons, the lengths of:

a) OM b) MX

Answers

a) $PM = MQ = \frac{1}{2}PQ = 8 \text{ cm}$ (XY is an axis of symmetry of the circle)

In ΔQPM :

$$OM^2 = OP^2 - PM^2 \quad \text{(Theorem of Pythagoras)}$$

$$= 10^2 - 8^2$$

$$= 100 - 64$$

$$\therefore OM = \sqrt{36}$$

$$\therefore OM = 6 \text{ cm}$$

b) $QX = QP$ (radius)

$= 10 \text{ cm}$

$$M_X = Q_X + Q_M$$

$$x = 10 + 6$$

$$= 10 + 6$$

Activity 1

1 In the diagram alongside, O is the centre of the circle.

a) If $OM = 5 \text{ cm}$ and $PQ = 24 \text{ cm}$, calculate, with reasons, the lengths of:

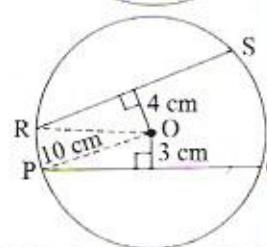
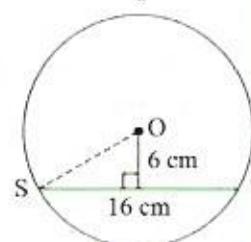
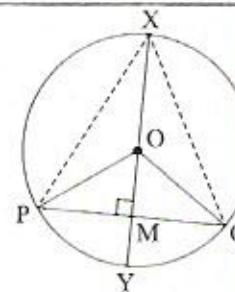
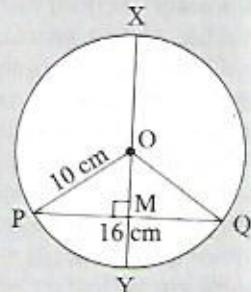
(i) OP

(ii) MX.

b) If $XM = 24 \text{ cm}$ and $PX = 25 \text{ cm}$, calculate, with reasons, the length of PO .

2 In the diagram alongside, O is the centre of the circle. Chord ST is 6 cm from the centre of the circle and the length of ST is 16 cm. Calculate, with reasons, the radius of the circle.

3 Two chords PQ and RS are 3 cm and 4 cm respectively from the centre of the circle, O, and the radius of the circle is 10 cm. Calculate, with reasons, the lengths of chords PQ and RS to 3 significant figures.



Introduction

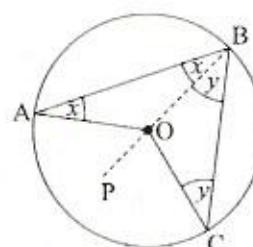
In this sub-topic, you will learn about the angle properties of circles, cyclic quadrilaterals and tangents to circles.

Solve problems using angle properties of a circle

You are now going to learn about a number of theorems that we use to solve problems involving circles.

Theorem 1

The angle which an arc subtends at the centre of a circle is twice that which it subtends at any point on the circumference.



Given: Circle with centre O and points A, B and C on the circumference of the circle.

Prove: $\angle AOC = 2 \times \angle ABC$

Construction: Produce BO to point P.

Proof: Let $\angle ABO = x$ and $\angle CBO = y$.

In $\triangle OAB$:

$OA = OB$ (radii)

$\therefore \angle OAB = x$ ($\triangle AOB$ is isosceles)

$\therefore \angle AOP = 2x$ (exterior angle of $\triangle AOB$)

Similarly, in $\triangle OBC$:

$\angle COP = 2y$

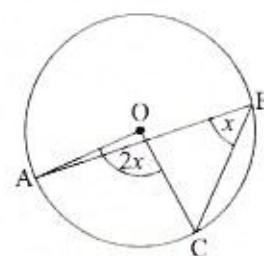
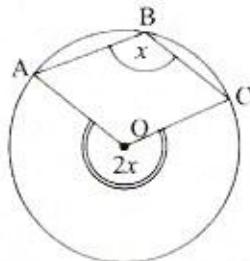
$\therefore \angle AOC = 2(x + y)$

$\therefore \angle AOC = 2 \times \angle ABC$

Take note of the position of the angles in the diagrams below. In each case, $\angle AOC = 2 \times \angle ABC$.

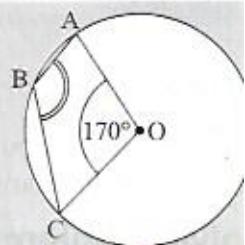
New words

subtend: be opposite to

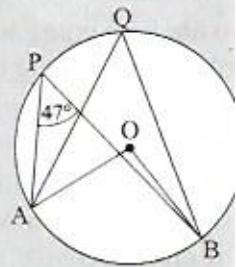


Worked example 2

1 In the diagram alongside, O is the centre of the circle and $\angle AOC = 170^\circ$. Calculate, with reasons, the size of $\angle ABC$.



2 In the diagram alongside, O is the centre of circle APQR and $\angle APB = 47^\circ$. Calculate, with reasons, the sizes of:
a) $\angle AOB$ b) $\angle AQB$.



Answers

1 Reflex $\angle AOC = 360^\circ - 170^\circ$ (angles around a point)
= 190°

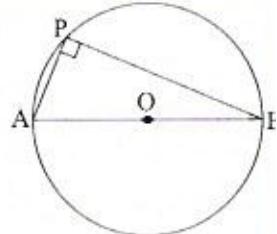
$2 \times \angle ABC = \text{reflex } \angle AOC$ (angle at centre is twice angle at circumference)
 $\therefore \angle ABC = \frac{190^\circ}{2}$
= 95°

2 a) $\angle AOB = 2 \times \angle APB$ (angle at centre is twice angle at circumference)
= $2 \times 47^\circ$
= 94°

b) $2 \times \angle AQB = \angle AOB$ (angle at centre is twice angle at circumference)
 $\therefore \angle AQB = \frac{94^\circ}{2}$
= 47°

Theorem 2

An angle in a semi-circle is a right angle.



Given: Circle ABP with centre O. AOB is a diameter of the circle and P is any point on the circumference.

Prove: $\angle APB = 90^\circ$

Proof: $\angle AOB = 2 \times \angle APB$ (angle at centre is twice angle at circumference)

$\angle AOB = 180^\circ$ (straight angle)

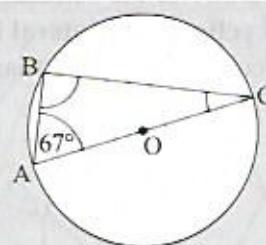
$\therefore 2 \times \angle APB = 180^\circ$

$\therefore \angle APB = 90^\circ$

Worked example 3

In the diagram alongside, O is the centre of the circle and $\angle BAC = 67^\circ$. Calculate, with reasons, the sizes of:

- $\angle ABC$
- $\angle ACB$.

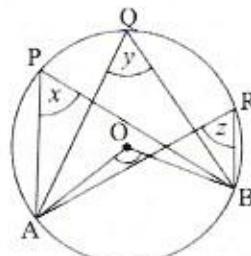


Answers

- $\angle ABC = 90^\circ$ (angle in a semi-circle)
- $\angle ACB = 180^\circ - (90^\circ + 67^\circ)$ (sum of angles of a triangle)
 $= 23^\circ$

Theorem 3

Angles in the same segment of a circle are equal.



Given: Circle with centre O and points A, P, Q, R and B on the circumference of the circle.

Prove: $\angle APB = \angle AQB = \angle ARB$

Proof: $\angle AOB = 2x$ (angle at centre is twice angle at circumference)

$\angle AOB = 2y$ (angle at centre is twice angle at circumference)

$\angle AOB = 2z$ (angle at centre is twice angle at circumference)

$$\therefore x = y = z = \frac{1}{2} \times \angle AOB$$

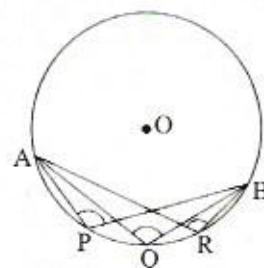
$$\therefore \angle APB = \angle AQB = \angle ARB$$

Note

$\angle APB$, $\angle AQB$ and $\angle ARB$ are all based on chord AB. We say that these angles are all subtended by chord AB.

In the diagram alongside, $\angle APB$, $\angle AQB$ and $\angle ARB$ are in the minor arc, but Theorem 3 holds true here as well:

$\angle APB = \angle AQB = \angle ARB$. These angles are also subtended by chord AB.



Solve problems involving cyclic quadrilaterals

A **cyclic quadrilateral** is a quadrilateral of which all four vertices lie on the circumference of the same circle. Below are two examples of cyclic quadrilaterals.

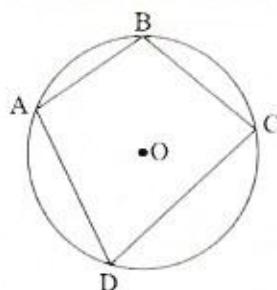


Figure 8.8

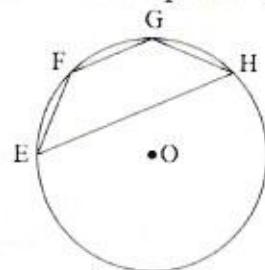


Figure 8.9

In Figure 8.8, ABCD is a **cyclic quadrilateral**. In Figure 8.9, EFGH is a cyclic quadrilateral.

New word

cyclic quadrilateral: a quadrilateral of which all four vertices lie on the circumference of the same circle

Theorem 4

The opposite angles of a cyclic quadrilateral are supplementary. Conversely, if the sum of a pair of opposite angles of a quadrilateral is 180° , the quadrilateral is cyclic.

Given: Cyclic quadrilateral ABCD in a circle with centre O.

Prove: $\angle ADC + \angle ABC = 180^\circ$

Construction: Join AO and CO.

Proof: Let $\angle ADC = x$ and $\angle ABC = y$.

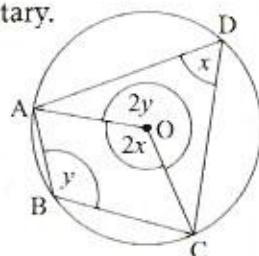
obtuse $\angle AOC = 2x$ (angle at centre is twice angle at circumference)

reflex $\angle AOC = 2y$ (angle at centre is twice angle at circumference)

$2x + 2y = 360^\circ$ (angles around a point)

$\therefore x + y = 180^\circ$

$\angle ADC + \angle ABC = 180^\circ$



So, for any cyclic quadrilateral PQRS, (see Figure 8.10), we have this result:

$x + y = 180^\circ$ and $w + z = 180^\circ$.

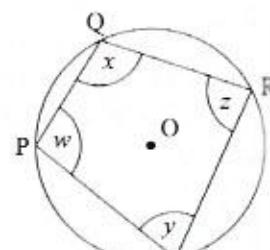
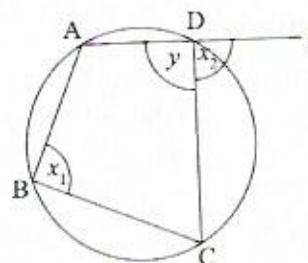


Figure 8.10

Theorem 5

The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.



Given: Cyclic quadrilateral ABCD with AD produced to point P.

Prove: $\angle ABC = \angle PDC$

Proof: $x_1 + y = 180^\circ$ (opposite angles of a cyclic quadrilateral)

$$\therefore x_1 = 180^\circ - y$$

$x_2 + y = 180^\circ$ (angles on a straight line)

$$\therefore x_2 = 180^\circ - y$$

$$\therefore x_1 = x_2$$

$$\therefore \angle ABC = \angle PDC$$

Worked example 4

1 In the diagram alongside, ABCD is a cyclic quadrilateral in a circle with centre O. $\angle ABD = 40^\circ$, $\angle BAC = 51^\circ$ and $\angle DAC = 18^\circ$.

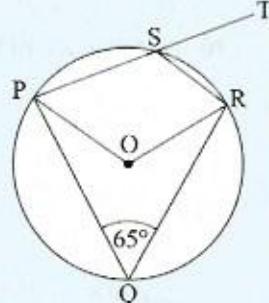
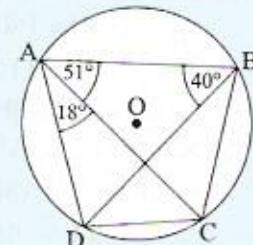
Calculate, with reasons, the sizes of:

- a) $\angle DBC$
- b) $\angle ACD$
- c) $\angle BDC$
- d) $\angle ACB$
- e) $\angle ADB$

2 In the diagram alongside, PQRS is a cyclic quadrilateral in a circle with centre O. PS has been produced to T and $\angle PQR = 65^\circ$.

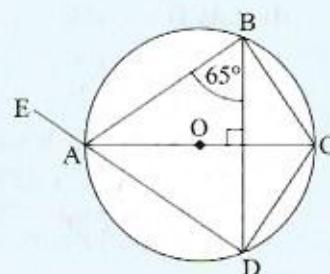
Calculate, with reasons, the sizes of:

- a) $\angle POR$
- b) $\angle PSR$
- c) $\angle TSR$



3 In the diagram alongside, ABCD is cyclic quadrilateral in a circle with centre O. AC is a diameter of the circle, $BD \perp AC$ and $\angle ABD = 65^\circ$. DA is produced to E. Calculate, with reasons, the sizes of:

- a) $\angle DBC$
- b) $\angle ACB$
- c) $\angle BDC$
- d) $\angle EAB$



Worked example 4 (continued)

Answers

1 a) $\angle DBC = \angle DAC$ (angles in the same segment)
 $\therefore \angle DBC = 18^\circ$

b) $\angle ACD = \angle ABD$ (angles in the same segment)
 $\therefore \angle ACD = 40^\circ$

c) $\angle BDC = \angle BAC$ (angles in the same segment)
 $\therefore \angle BDC = 51^\circ$

d) $\angle ACB = 180^\circ - (51^\circ + 40^\circ + 18^\circ)$ (angles in a triangle add up to 180°)
 $= 71^\circ$

e) $\angle ADB = \angle ACB$ (angles in the same segment)
 $\therefore \angle ADB = 71^\circ$

2 a) $\angle POR = 2 \times \angle PQR$ (angle at centre is twice angle at circumference)
 $= 2 \times 65^\circ$
 $= 130^\circ$

b) $\angle PSR + \angle PQR = 180^\circ$ (opposite angles of a cyclic quadrilateral)
 $\therefore \angle PSR = 180^\circ - 65^\circ$
 $= 115^\circ$

c) $\angle TSR = \angle PQR$ (exterior angle of a cyclic quadrilateral)
 $\therefore \angle TSR = 65^\circ$

3 a) $\angle DBC + \angle ABD = 90^\circ$ (angle in a semi-circle)
 $\therefore \angle DBC = 90^\circ - 65^\circ$
 $= 25^\circ$

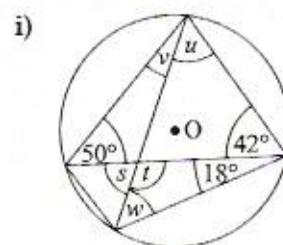
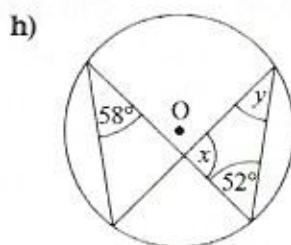
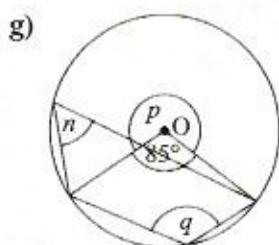
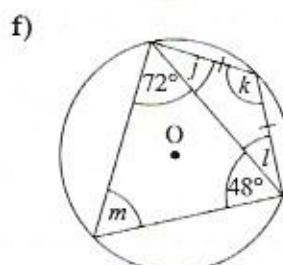
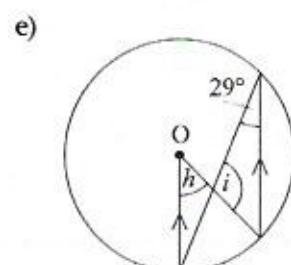
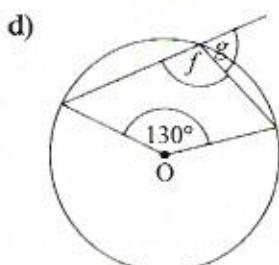
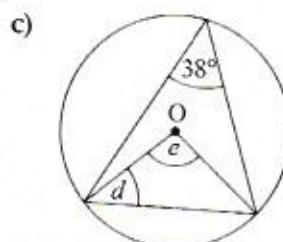
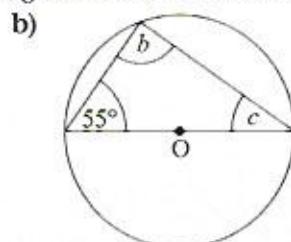
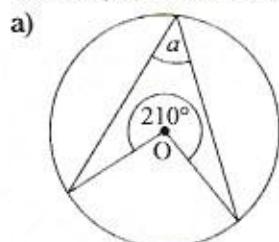
b) $\angle ACB + \angle CBD + 90^\circ = 180^\circ$ (sum of angles of a triangle)
 $\angle ACB = 180^\circ - (25^\circ + 90^\circ)$
 $= 65^\circ$

c) $\angle BDC = \angle BAC$ (angles in the same segment)
 $\angle BAC = 180^\circ - (90^\circ + 65^\circ)$ (sum of angles of a triangle)
 $= 25^\circ$
 $\therefore \angle BDC = 25^\circ$

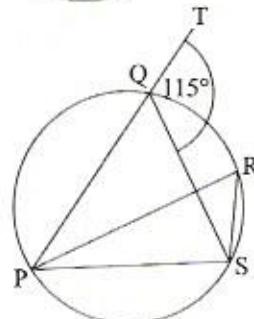
d) $\angle ACD = \angle ABD$ (angles in the same segment)
 $\therefore \angle ACD = 65^\circ$
 $\angle EAB = \angle BCD$ (exterior angle of a cyclic quadrilateral)
 $\therefore \angle EAB = \angle ACB + \angle ACD$
 $\therefore \angle EAB = 65^\circ + 65^\circ$
 $= 130^\circ$

Activity 2

1 In each of the diagrams below, O is the centre of the circle. Calculate, with reasons, the sizes of the angles marked with letters.

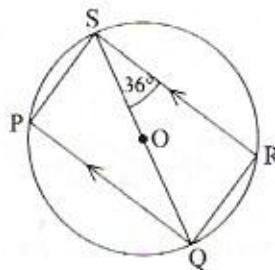


2 In the diagram on the right, PQRS are points on a circle. PQ is a straight line produced to T and $\angle TQR = 115^\circ$. Calculate, with reasons, the size of $\angle PRS$.



3 PQRS is a cyclic quadrilateral in a circle with centre O. PQ is parallel to SR and $\angle QSR = 36^\circ$. Calculate, with reasons, the sizes of:

a) $\angle QRS$ b) $\angle SQR$
c) $\angle PQS$

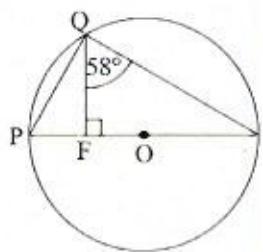


Activity 2 (continued)

4 PQR is a circle with centre O and $\angle RQF = 58^\circ$.

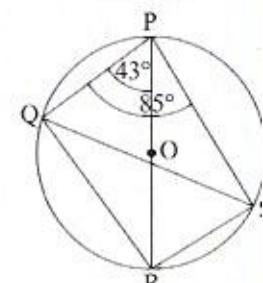
Calculate, with reasons, the sizes of:

- $\angle PQF$
- $\angle QPF$



5 PQRS is a cyclic quadrilateral in a circle with centre O. $\angle QPR = 43^\circ$ and $\angle QPS = 85^\circ$. Calculate, with reasons, the sizes of:

- $\angle QRS$
- $\angle SQR$



Solve problems involving tangent properties

When you have a straight line and a circle, there are three possibilities.

- The line does not cut or touch the circle. See Figure 8.11.
- The line cuts the circle in two points. See Figure 8.12.
- The line touches the circle at one point only. See Figure 8.13.

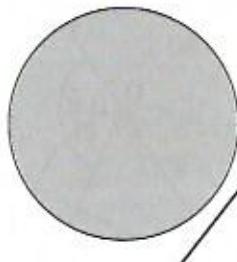


Figure 8.11

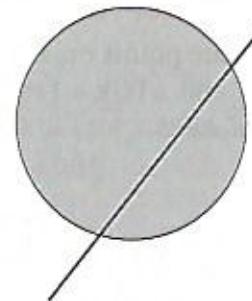


Figure 8.12

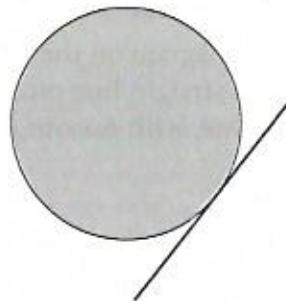


Figure 8.13

A line that cuts a circle in two points is called a **secant**. A line that touches a circle at one point only is called a **tangent**.

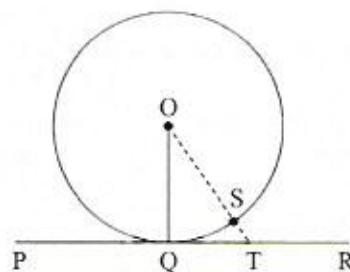
Note

secant: a line that cuts a circle in two points

tangent: a line that touches a circle at one point only

Theorem 6

A tangent to a circle is perpendicular to the radius at the point of contact.



Given: Circle with centre O and tangent PQR, where Q is the point of contact with the circle.

Prove: $\angle OQR = 90^\circ$

Construction: Draw OST, with T on QR and S on the circumference of the circle.

Proof: Assume that OQ is not perpendicular to PQR. Then there must be another point, say T, on PQR such that $OT \perp PQR$.

If $\angle OTQ = 90^\circ$, then $\triangle OTQ$ is a right-angled triangle, with OQ the hypotenuse of the triangle.

$\therefore OQ > OT$

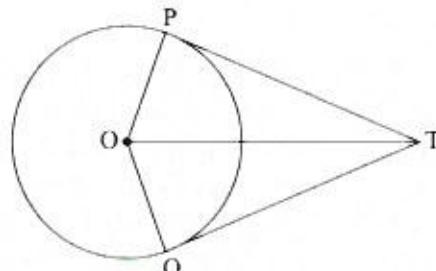
But this is impossible, because $OQ = OS$ (radii).

$\therefore OQ < OT$

$\therefore \angle OQR = 90^\circ$

Theorem 7

The tangents to a circle from an external point are equal.



Given: A point T outside a circle with centre O. TP and TQ are tangents to the circle at P and Q.

Prove: $TP = TQ$

Construction: Join OP, OQ and OT.

Proof: In $\triangle OPT$ and $\triangle OQT$:

$\angle OPT = \angle OQT = 90^\circ$ (radius \perp tangent)

$OP = OQ$ (radii)

$OT = OT$ (common side)

$\therefore \triangle OPT \cong \triangle OQT$ (RHS)

$\therefore TP = TQ$

Alternate segments

In Figure 8.14 and Figure 8.15 below, RBT is a tangent to each circle at B. Chord AB divides each circle into two segments, ABQ and ABP.

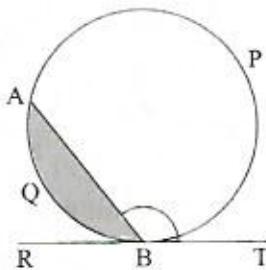


Figure 8.14

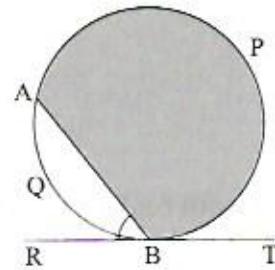


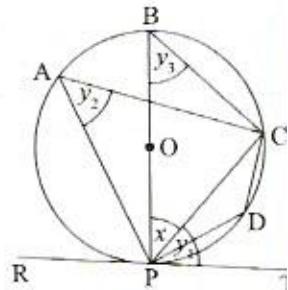
Figure 8.15

In Figure 8.14, the segment AQB is the alternate segment to $\angle TBA$.

In Figure 8.15, the segment APB is the alternate segment to $\angle RBA$.

Theorem 8

If a chord to a circle is drawn from the point of contact at which the tangent touches the circle, then the angle which the tangent makes with the chord is equal to any angle in the alternate segment.



Given: Circle with centre O, tangent RPT and chord PC dividing the circle into segments PABC and PDC.

Prove: $\angle TPC = \angle PAC$

Construction: Draw diameter BP and join BC.

Proof: $x + y_1 = 90^\circ$ (tangent \perp radius)

$$\therefore y_1 = 90^\circ - x$$

$\angle PCB = 90^\circ$ (angle in semi-circle)

$\therefore x + y_3 = 180^\circ - 90^\circ$ (sum of angles of a triangle)

$$\therefore x + y_3 = 90^\circ$$

$$\therefore y_3 = 90^\circ - x$$

$$\therefore y_1 = y_3$$

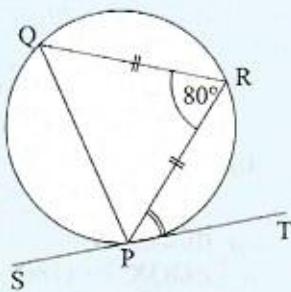
$y_2 = y_3$ (angles in the same segment)

$$\therefore \angle TPC = \angle PAB$$

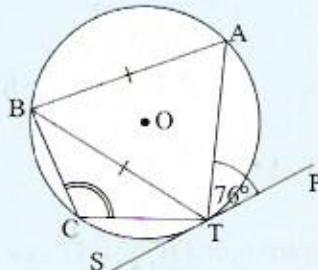
Similarly, it can be proven that $\angle RPC = \angle PDC$.

Worked example 5

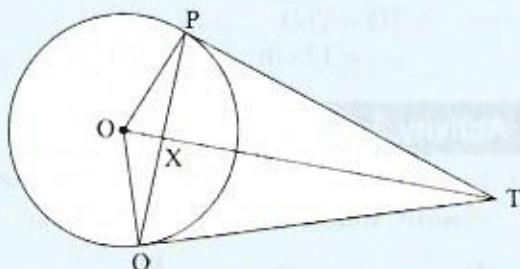
1 In the diagram alongside, PQ, PR and RQ are chords of a circle. SPT is a tangent to the circle at P and $\angle QRP = 80^\circ$. Calculate, with reasons, the size of $\angle RPT$.



2 In the diagram alongside, ABCT is a circle, STP is a tangent to the circle at T and $\angle ATP = 76^\circ$. Calculate, with reasons, the size of $\angle BCT$.



3 In the diagram alongside, O is the centre of the circle and PT and QT are tangents to the circle.



a) If $\angle POX = 61^\circ$, calculate, with reasons, the size of $\angle QTX$.
 b) If $\angle POT = 58^\circ$, calculate, with reasons, the size of $\angle PQT$.
 c) If $OQ = 5$ cm and $OT = 13$ cm, calculate, with reasons, the length of TQ.

Answers

1 $\angle RQP = \frac{1}{2}(180^\circ - 80^\circ)$ (sum of angles of an isosceles triangle)
 $= 50^\circ$

$\angle RPT = \angle RQP$ (angles in alternate segments)
 $\therefore \angle RPT = 50^\circ$

2 $\angle ABT = \angle ATP$ (angles in alternate segments)
 $\therefore \angle ABT = 76^\circ$

$\angle BAT = \frac{1}{2}(180^\circ - 76^\circ)$ (sum of angles of an isosceles triangle)
 $= 52^\circ$

$\angle BCT = 180^\circ - 52^\circ$ (opposite angles of a cyclic quadrilateral)
 $= 128^\circ$

Worked example 5 (continued)

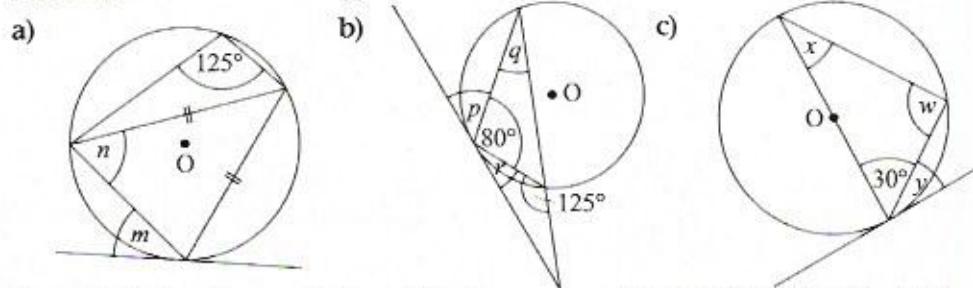
3 a) $\angle OPT = 90^\circ$ (radius \perp tangent)
 $\therefore \angle PTO = 180^\circ - (90^\circ + 61^\circ)$ (sum of angles of a triangle)
 $= 29^\circ$
 $\triangle OPT \cong \triangle OQT$ (RHS) (proved in Theorem 7)
 $\therefore \angle QTO = \angle PTO$
 $\therefore \angle QTX = 29^\circ$

b) $\angle QOT = \angle POT$ ($\triangle OPT \cong \triangle OQT$)
 $\therefore \angle QOT = 58^\circ$
In $\triangle OPQ$:
 $\angle OQX = \frac{1}{2}(180^\circ - 58^\circ - 58^\circ)$ (sum of angles of an isosceles triangle)
 $= 32^\circ$
But $\angle OQT = 90^\circ$ (radius \perp tangent)
 $\therefore \angle PQT = 90^\circ - 58^\circ$
 $= 32^\circ$

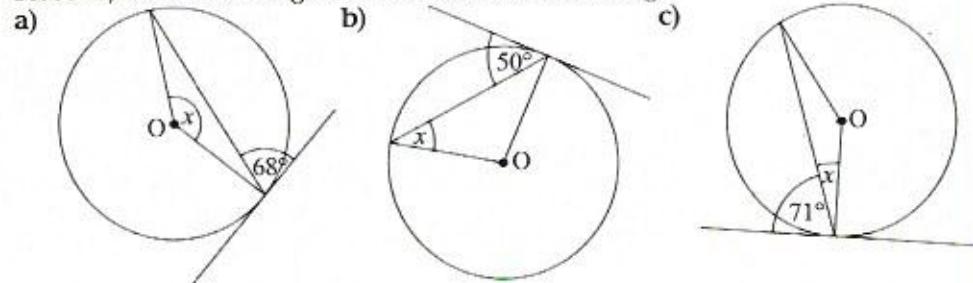
c) $TQ^2 = OT^2 - OQ^2$ (Theorem of Pythagoras)
 $= 13^2 - 5^2$
 $= 169 - 25$
 $= 144$
 $\therefore TQ = \sqrt{144}$
 $= 12 \text{ cm}$

Activity 3

1 In each of the diagrams below, O is the centre of the circle. Calculate, with reasons, the sizes of the angles marked with letters.

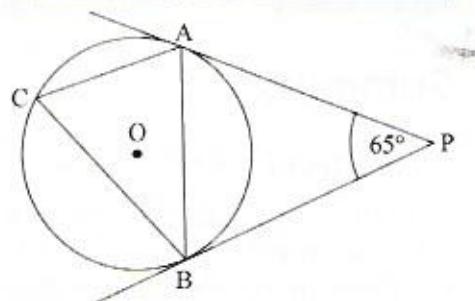


2 In each of the diagrams below, O is the centre of the circle. Calculate, with reasons, the size of angle x in each of the following.

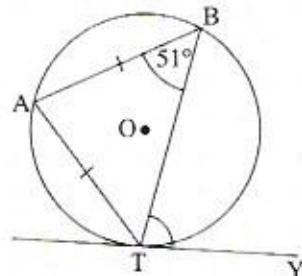


Activity 3 (continued)

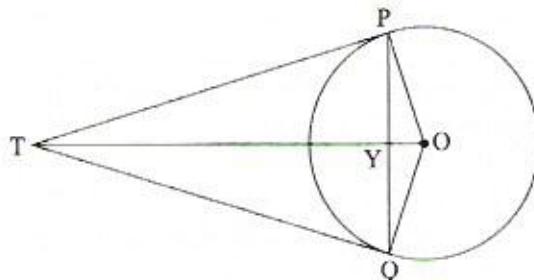
3 A, B, C are three points on a circle with centre O. Two tangents through A and B meet at P. $\angle APB = 65^\circ$. Calculate, with reasons, the size of $\angle ACB$.



4 In the diagram alongside, TY is a tangent to circle ABT. $\angle ABT = 51^\circ$ and $AT = AB$. Calculate, with reasons, the size of $\angle BTY$.



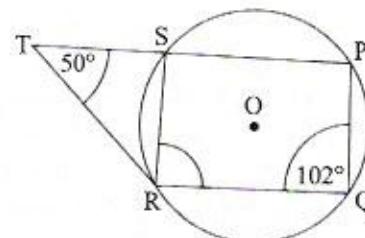
5 In the diagram below, O is the centre of the circle and TP and TQ are tangents to the circle.



- If $\angle POT = 48^\circ$, calculate, with reasons, the size of $\angle PQO$.
- If $\angle QTO = 46^\circ$, calculate, with reasons, the size of $\angle TPY$.
- If $\angle PTO = 51^\circ$, calculate, with reasons, the size of $\angle PQO$.
- If $PQ = 12$ cm and $TQ = 10$ cm, calculate, with reasons, the length of TY .
- Calculate, with reasons, the length of TO , if $TP = 12$ cm and $PO = 9$ cm.
- Calculate, with reasons, the size of $\angle TQY$, if $\angle PTO = 46^\circ$.

6 P, Q, R, S are points on a circle.

$\angle PQR = 102^\circ$. PS is produced to T, such that $\angle STR = 50^\circ$. Calculate, with reasons, the size of $\angle SRT$.



Summary

Properties of a circle

- A chord is a straight line drawn across a circle that joins any two points on its circumference.
- A chord that passes through the centre of a circle is called a diameter.
- A line drawn from the centre of a circle to its circumference is called a radius. A diameter consists of two radii.
- A chord which is not a diameter divides the circumference into a major arc and a minor arc.
- A chord divides a circle into a major segment and a minor segment.
- A diameter divides the circle into two equal halves. These are called semi-circles.
- A sector of a circle is an area bounded by two radii and an arc. The two radii divide the circle into a major sector and a minor sector.
- A diameter (or radius) drawn perpendicular to a chord bisects the chord.
- The perpendicular bisector of a chord passes through the centre of a circle.
- Equal chords are equidistant from the centre of the circle.

Angle properties

- The angle which an arc subtends at the centre of a circle is twice that which it subtends at any point on the circumference. (Theorem 1)
- An angle in a semi-circle is a right angle. (Theorem 2)
- Angles in the same segment of a circle are equal. (Theorem 3)
- A cyclic quadrilateral is a quadrilateral of which all four vertices lie on the circumference of the same circle.
- The opposite angles of a cyclic quadrilateral are supplementary. Conversely, if the sum of a pair of opposite angles of a quadrilateral is 180° , the quadrilateral is cyclic. (Theorem 4)
- The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle. (Theorem 5)
- A line that cuts a circle in two points is called a secant.
- A line that touches a circle at one point only is called a tangent.
- A tangent to a circle is perpendicular to the radius at the point of contact. (Theorem 6)
- The tangents to a circle from an external point are equal. (Theorem 7)
- If a chord to a circle is drawn from the point of contact of a tangent, then the angle which the tangent makes with the chord is equal to any angle in the alternate segment. (Theorem 8)

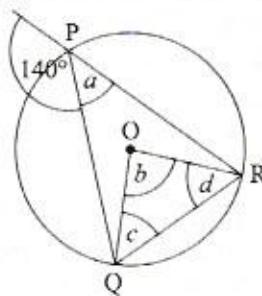
Revision exercises

3 In each of the diagrams below, O is the centre of the circle.

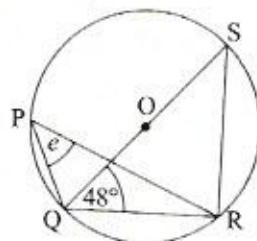
Calculate, with reasons, the sizes of the angles marked with letters.

(26)

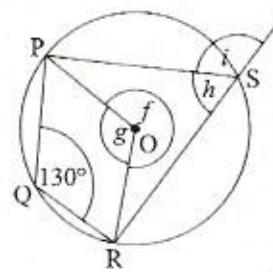
a)



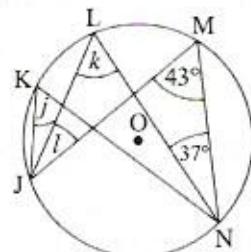
b)



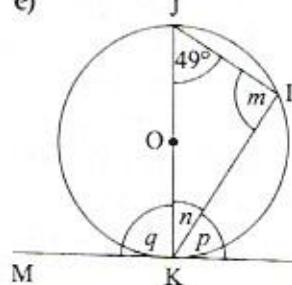
c)



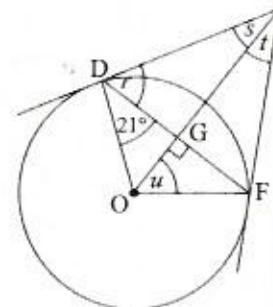
d)



e)



f)



4 In the diagram alongside, P, Q, R and S are points on a circle with centre O and QS as diameter. TS is a tangent at S and $\angle TSP = 40^\circ$. Calculate, with reasons, the sizes of:

a) $\angle PSQ$

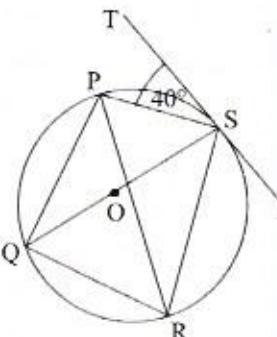
(2)

b) $\angle PQS$

(2)

c) $\angle PRS$

(2)



5 In the diagram alongside, P, Q, R are points on a circle with centre O. $\angle POR = 125^\circ$ and $\angle ORQ = 31^\circ$.

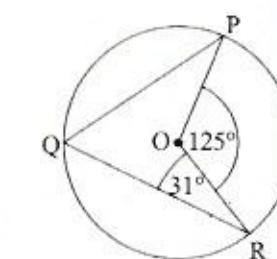
Calculate, with reasons, the sizes of:

a) $\angle PQR$

(2)

b) $\angle QPO$

(4)

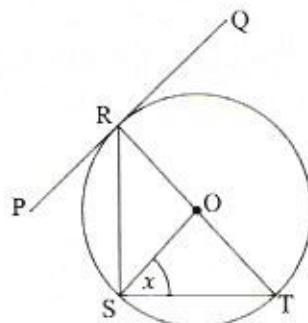


Total marks: 60

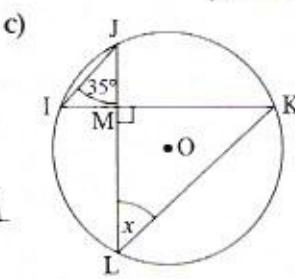
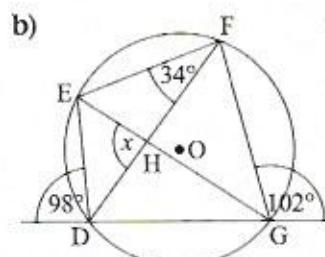
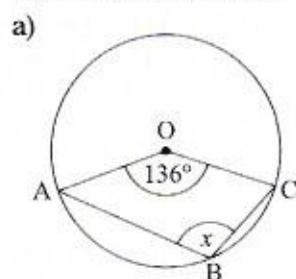
Assessment exercises

1 In the diagram alongside, PQ is a tangent to the circle with centre O . $\angle OST = x$. Express each of the following angles in terms of x .

- a) $\angle OTS$ (1)
- b) $\angle SOT$ (1)
- c) $\angle RSO$ (1)
- d) $\angle PRS$ (1)

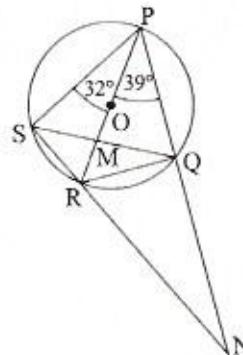


2 Calculate, with reasons, the size of angle x in each of the following circles with centre O . $(3 \times 4 = 12)$



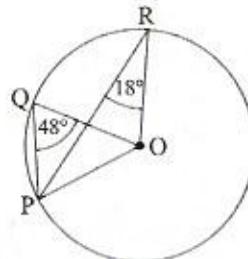
3 In the diagram alongside, PR is a diameter of the circle $PQRS$. PR and SQ intersect at M . PQ and SR are produced to N . Calculate, with reasons, the sizes of:

- a) $\angle QRS$ (2)
- b) $\angle PRS$ (4)
- c) $\angle PQM$ (2)
- d) $\angle PMQ$ (2)
- e) $\angle PNS$ (4)



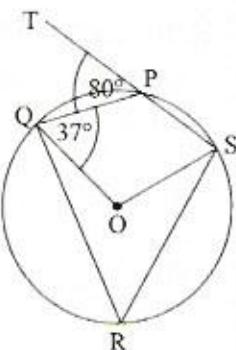
4 In the diagram alongside, P , Q , R are points on a circle with centre O . $\angle PZO = 48^\circ$ and $\angle PRO = 18^\circ$.

Calculate, with reasons, the size of $\angle ROQ$. (6)



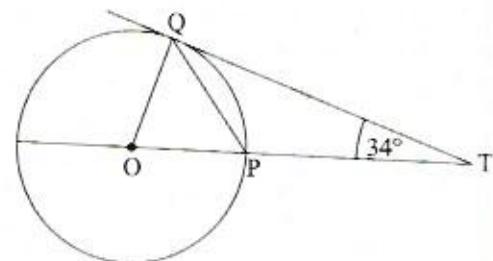
Revision and assessment continued

5 In the diagram alongside, P, Q, R, S are points on a circle with centre O. SP is produced to T. $\angle P Q O = 37^\circ$ and $\angle T P Q = 80^\circ$. Calculate, with reasons, the size of $\angle P S O$.



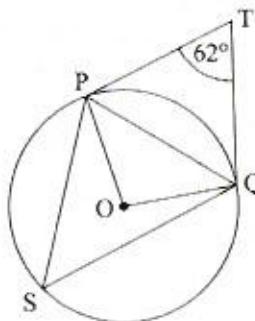
(8)

6 In the diagram alongside, TQ is a tangent at Q to the circle with centre O and $\angle O T Q = 34^\circ$. Calculate, with reasons, the size of $\angle Q P T$.



(8)

7 In the diagram below, P, Q and S are points on a circle with centre O. TP and TQ are tangents. $\angle P T Q = 62^\circ$.



Calculate, with reasons, the sizes of:

- a) $\angle P S Q$
- b) $\angle P O Q$
- c) $\angle O Q P$

(4)

(2)

(2)

Total marks: 60



Sub-topics	Specific Outcomes
Constructions	<ul style="list-style-type: none">Construct line and angle bisectors
The locus of a point	<ul style="list-style-type: none">Explain the meaning of a locus
Loci in two dimensions	<ul style="list-style-type: none">Describe the locus of a point in two dimensionsConstruct the locus of a point in two dimensions
Loci in three dimensions	<ul style="list-style-type: none">Describe the locus of a point in three dimensions

Starter activity

Work in pairs for this activity.

On the right is a photograph of the Victoria Falls Bridge. Many people worked together to design and build this bridge.

- 1 This bridge is situated on the border of Zambia and another country. Name the country.
- 2 Name at least three different kinds of professionals who were involved in designing and building this bridge.
- 3 This bridge is used to carry three different kinds of transport. What are they? Find out, if you do not know.
- 4 Name at least three different geometrical shapes that you can see in the photograph.



Victoria Falls Bridge

SUB-TOPIC 1 Constructions

A geometrical construction is an accurate drawing, using geometrical instruments. In real life, the word "construction" usually refers to something that is built, for example a building, a road or a bridge. Many people need to be able to do accurate mathematical constructions in real life. These include architects, inventors, designers and engineers.

Below are some useful constructions:

Description	Steps to follow	Diagram
To bisect a line	<ul style="list-style-type: none"> Draw any line AB. Set the pair of compasses to any length greater than half of AB. With centre A, construct arcs above and below line AB. With centre B, construct arcs to cut the arcs in the previous step at points X and Y respectively. The line through X and Y is the perpendicular bisector of AB. 	
To bisect an angle	<ul style="list-style-type: none"> Draw any $\angle ABC$. With centre B and any radius, construct an arc to cut AB at X and BC at Y. With centres X and Y, and using a large enough radius, construct arcs to intersect at P. The straight line through BP bisects $\angle ABC$. 	

New words

bisect: divide into two equal halves

perpendicular bisector: a line that bisects another line at an angle of 90°

Activity 1

- 1 a) Construct $\triangle ABC$ with $AB = 8.5$ cm, $BC = 6$ cm and $AC = 7$ cm.
b) Bisect lines AB and BC .
- 2 a) Construct equilateral $\triangle LMN$ with sides of 5 cm.
b) Bisect $\angle MLN$ and LM .
- 3 a) Construct $\triangle RST$ with $RS = 9$ cm, $ST = 7$ cm and $RT = 6$ cm.
b) Bisect RS , ST and RT . Label the point where the bisectors of the lines meet as O .
c) Draw a circle with centre O and radius OR . What do you notice about this circle?

SUB-TOPIC 2

The locus of a point

The **locus** of a point is the path along which the point may move in order to satisfy one or more conditions. **Loci** is the plural of locus.

New words

locus: the path along which a point may move in order to satisfy one or more conditions

loci: the plural of locus

equidistant (from): equally far from, the same distance from

Below are some examples of loci:

Description	Solution	Diagram
The locus of a point which moves in such a way that it is always equidistant from a fixed point	The locus is the circumference of the circle with centre O.	
The locus of a point which moves in such a way that it is equidistant from two intersecting straight lines, AB and AC	The locus is the bisector AD of ∠BAC.	
The locus of a point P which moves in such a way that the area of ΔAPB is constant	The locus is an infinitely long straight line through the points Q, P and R, parallel to AB.	

Activity 2

Work with a partner for this activity.

- 1 Study the examples of loci above. Make sure that both of you understand each one.
- 2 a) Why are all points on the circumference of a circle equidistant from the centre of the circle?
b) Why are all points on the bisector of an angle equidistant from the lines that form the angle?
c) Why, for all points on the third locus, is the area of ΔAPB constant?

In this sub-topic, you will describe and construct loci in two dimensions. A locus in two dimensions is the path along which a point can move in a two-dimensional plane.

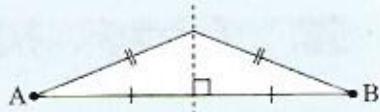
Describe the locus of a point in two dimensions

Worked example 1

1 In the diagram alongside, the locus is the perpendicular bisector of the straight line joining A and B. Describe this locus.

2 A point P may move anywhere outside a circle with radius 6.5 cm, as long as it is always exactly 2.5 cm from the circumference. Describe the locus of point P.



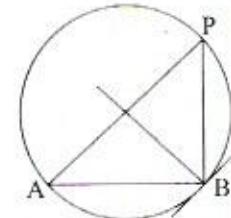
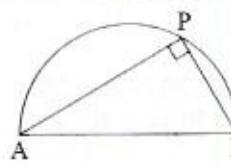


Answers

- 1 The point moves in such a way that it is always equidistant from points A and B.
- 2 The locus of point P is a circle with the same centre as the given circle, but with a radius of 9 cm.

Activity 3

Work with a partner for this activity.



Activity 3 (continued)

- 3 A point P may move anywhere within a circle with radius 5 cm, as long as it is always exactly 1.5 cm from the circumference. Describe the locus of point P.
- 4 ABCD is a cyclic quadrilateral with points A, B and C marked on the circumference of a circle. Describe the locus of point D.
- 5 XY is a straight line. Describe the locus of point Z, if $\triangle XYZ$ is:
 - a) isosceles, with $XZ = YZ$
 - b) equilateral
 - c) right-angled, with $\angle XYZ = 90^\circ$.

Construct the locus of a point in two dimensions

You will now use your knowledge of constructions to construct loci.

Activity 4

Work in pairs for this activity.

Show all construction marks. After you have completed each construction, check one another's work to make sure that you have done the construction correctly.

- 1 Construct the locus of a point that is 4 cm from a fixed point.
- 2 Construct the locus of a point that is 2 cm from a line that is 8 cm in length.
- 3 Construct the locus of a point that is equidistant from PQ and QR where $\angle PQR = 60^\circ$.
- 4 Construct the locus of a point B such that $\angle ABC = 90^\circ$ and $AC = 6$ cm.
- 5 Construct the locus of a point that is equidistant from A and B, where $AB = 5$ cm.
- 6 Construct the locus of a point such that the area of $\triangle ABC = 12$ cm^2 and $AB = 6$ cm.
- 7 a) Construct $\triangle ABC$ with base AB, where $AB = 8.5$ cm, $\angle BAC = 45^\circ$, $\angle ABC = 60^\circ$ and C is above AB.
 b) Construct the locus of a point within $\triangle ABC$ that is equidistant from AB and BC.
 c) Construct the locus of a point within $\triangle ABC$ that is 3 cm from B.
 d) Show, by shading, the region which contains points less than 3 cm from B and nearer to BC than to BA.
- 8 a) Construct $\triangle PQR$ with base PQ, where $PQ = 6.5$ cm, $PR = 10$ cm, $QR = 5$ cm and R is below PQ.
 b) Construct the locus of a point within $\triangle PQR$ that is 2.5 cm from Q.
 c) Construct the locus of a point within $\triangle PQR$ that is equidistant from QR and QP.
 d) Mark the point X which lies on both loci in Questions b) and c).
- 9 a) Prove that a triangle with sides $AC = 10$ cm, $BC = 6$ cm and $AB = 8$ cm contains a right angle.
 b) With base AC and B above AC, construct $\triangle ABC$ as described in Question a).
 c) Construct the locus of a point P such that $\angle APC = 90^\circ$, with P on the same side of AC as B.

SUB-TOPIC 4 Loci in three dimensions

In this sub-topic, you will describe and construct loci in three dimensions. A locus in three dimensions is the path along which a point can move in three-dimensional space.

Describe and construct the locus of a point in three dimensions

Worked example 2

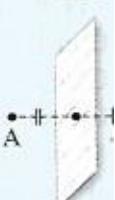
- 1 Describe and construct the locus of a point P that is 4 cm from a fixed point Q in three-dimensional space.
- 2 Describe and construct the locus of a point P that is equidistant from two points, A and B, in three-dimensional space.

Answers

- 1 The locus of P is the surface of a sphere with centre Q and radius 4 cm, as shown alongside.



- 2 The locus of P is the infinite plane that lies on the perpendicular bisector of the line AB, as shown alongside.



Activity 5

Work in small groups for this activity. If you struggle to construct a locus, you can support your drawing by naming a real-life object that has the shape of the locus.

- 1 Describe and construct the locus of a point P that is on the outside of a cube with edges 7 cm and is 2 cm from the closest face of the cube.
- 2 Given a circle with centre O, describe and construct the locus of a point P such that P is the apex of a right circular cone, with the circle the base of the cone. (Hint: A right circular cone is a cone of which the apex is directly above the centre of the circular base.)
- 3 Describe and construct the locus of a point P that is 10 cm from an infinite long line in three-dimensional space.
- 4 Given a circle with centre O and radius 25 cm, describe and construct the locus of a point P that is 3 cm from the circumference of the circle in three-dimensional space.

Summary

Constructions

- A geometrical construction is an accurate drawing, using geometrical instruments.

The locus of a point

- The locus of a point is the path along which the point may move in order to satisfy one or more conditions. Loci is the plural of locus.

Loci in two dimensions

- A locus in two dimensions is the path along which a point can move in a two-dimensional plane.

Loci in three dimensions

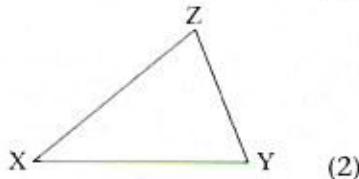
- A locus in three dimensions is the path along which a point can move in three-dimensional space.

Revision exercises (remedial)

- 1 a) Draw a line of 9 cm and label the endpoints A and B. (1)
b) Bisect line AB. (2)
- 2 a) Construct an angle of 50° and label it $\angle LMN$. (2)
b) Bisect $\angle LMN$. (2)

Revision exercises

- 3 a) Construct $\triangle PQR$ with $PQ = 8$ cm, $QR = 5$ cm and $PR = 6.5$ cm. (3)
b) Bisect line PQ . c) Bisect $\angle PRQ$. (2 x 2)
- 4 a) Construct an equilateral $\triangle DEF$ with sides of 5.7 cm. (3)
b) Bisect $\angle DEF$. c) Bisect DE . (2 x 2)
- 5 In the diagram below, Z is a vertex of $\triangle XYZ$.
a) Describe the locus of point Z, if the length of ZY must remain constant and $\triangle XYZ$ must remain a triangle. (2)
b) Describe the locus of point Z, if the area of $\triangle XYZ$ must remain constant.



Total marks: 25

Assessment exercises

- 1 a) Construct $\triangle XYZ$ with $XY = 5$ cm, $YZ = 6$ cm and $XZ = 7$ cm. (3)
b) Bisect all three angles of the triangle. Label the point where the bisector cuts YZ as P . Label the point where the bisectors meet as O . (6)
c) Draw a circle with centre O and radius OP . (2)
d) What do you notice about this circle? (2)
- 2 Consider two points, A and C , in two-dimensional space. Points B and D are both equidistant from A and C .
a) Describe the shared locus of points B and D . (2)
b) If points B and D fall on different points on their shared locus, what kind of quadrilateral is $ABCD$? (1)
- 3 Line AB is 6 cm long.
a) Describe the locus of point P , if it is equidistant from A and B . (2)
b) Construct the locus in Question 3a. (2)
c) Describe the locus of point Q if it is 5 cm from M , the midpoint of A and B . (2)
d) Construct the locus in Question 3c. (2)
e) Describe the locus of point R , if it is 2 cm from line AB . (3)
f) Construct the locus in Question 3e. (3)
- 4 The locus of point X is such that it is always 5 cm from point A . The locus of point Y is such that it is always 3 cm from point B .
a) Describe the loci of points X and Y . (4)
b) If A and B are placed in such a way that the loci do not touch or overlap, describe all the possible distances between points A and B . (6)
Draw one or more diagrams to support your answer.
- 5 An infinitely long line stretches out into space. The locus of point P is such that it is never more than 2 cm from the line.
a) Describe the locus of point P . (2)
b) Construct the locus in Question 5a. (2)
- 6 A wheelbarrow is pushed in a straight line up a hill. If the diameter of the wheel of the wheelbarrow is 30 cm, describe the locus of the centre of the wheel. (2)
- 7 A 2-m high ladder is placed upright against a wall. It is slowly allowed to slide down the wall until it comes to rest flat on the ground. The top of the ladder touches the wall throughout this process. Describe the locus of the midpoint of the ladder.



Total marks: 50



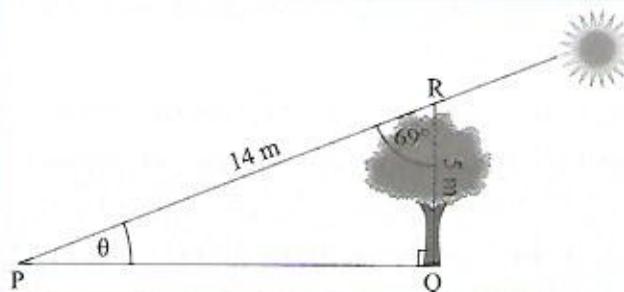
Sub-topics	Specific Outcomes
Introduction to trigonometry	<ul style="list-style-type: none"> Relate right-angled triangles to the three trigonometric ratios
Trigonometric ratios	<ul style="list-style-type: none"> Describe the three trigonometric ratios on a right-angled triangle Calculate sides and angles of a right-angled triangle Work with special angles ($30^\circ, 45^\circ, 60^\circ$)
Sine and cosine rules	<ul style="list-style-type: none"> Find sides and angles of non right-angled triangles
Area of triangles	<ul style="list-style-type: none"> Calculate the area of non right-angled triangles
Trigonometry on the Cartesian plane	<ul style="list-style-type: none"> Determine the signs of the three trigonometric ratios in the quadrants Draw graphs for sine, cosine and tangent curves Solve trigonometric equations
Applications of trigonometry	<ul style="list-style-type: none"> Use trigonometry to solve practical problems

Starter activity

Work in pairs for this activity.

In the diagram alongside, PQ is the length of the shadow cast by a tree, QR , that is 5 m tall.

- Find the value of θ .
- Calculate the length of the shadow, correct to the nearest centimetre.
- If the tree had been 10 m tall, what would the length of its shadow have been, correct to the nearest centimetre? Explain your reasoning.



The word trigonometry comes from two Greek words: *trigonon*, which means 'three angles', and *metro*, which means 'measure'. So trigonometry deals with the measurements of triangles. Trigonometry is used in many fields in real life, including architecture, engineering, land surveying, map-making, navigation and astronomy.

New word

trigonometry: the branch of mathematics that deals with the measurements of triangles

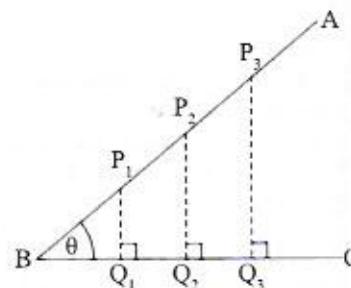
In this sub-topic, we will explore some ideas that form the basis of trigonometry.

Relate right-angled triangles to the three trigonometric ratios

In the diagram alongside, $\angle ABC = \theta$. P_1 , P_2 and P_3 are points on straight line AB . From each point P_1 , P_2 and P_3 , perpendiculars have been drawn to the line BC .

There are three similar triangles in the diagram alongside. These are: $\triangle BP_1Q_1$, $\triangle BP_2Q_2$ and $\triangle BP_3Q_3$. Note that $\angle ABC$ is common to all three triangles. Since we have similar triangles, it follows that the lengths of corresponding sides are proportional. So, we have:

$$\bullet \frac{P_1Q_1}{BP_1} = \frac{P_2Q_2}{BP_2} = \frac{P_3Q_3}{BP_3}$$



This means that, for a given $\angle ABC$, there is only one value of $\frac{PQ}{BP}$ for all positions of P on AB .

$$\bullet \frac{BQ_1}{BP_1} = \frac{BQ_2}{BP_2} = \frac{BQ_3}{BP_3}$$

Similarly, for a given $\angle ABC$, there is only one value of $\frac{BQ}{BP}$ for all positions of P on AB .

$$\bullet \frac{P_1Q_1}{BQ_1} = \frac{P_2Q_2}{BQ_2} = \frac{P_3Q_3}{BQ_3}$$

Finally, for a given $\angle ABC$, there is only one value of $\frac{PQ}{BQ}$ for all positions of P on AB .

Therefore, for any $\angle ABC$, there is only one value of each of the ratios $\frac{PQ}{BP}$, $\frac{BQ}{BP}$ and $\frac{PQ}{BQ}$ for all the positions of P on AB .

These ratios have special names in trigonometry. We say that

$$\frac{PQ}{BP} = \sin \angle ABC,$$

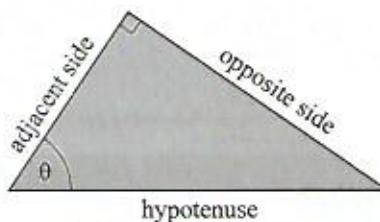
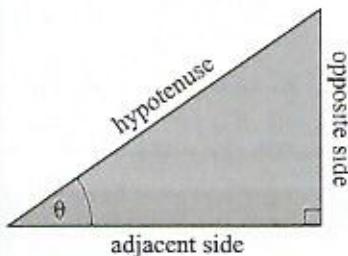
$$\frac{BQ}{BP} = \cos \angle ABC$$

$$\text{and } \frac{PQ}{BQ} = \tan \angle ABC.$$

You will learn more about these ratios in the next sub-topic.

Describe the three trigonometric ratios on a right-angled triangle

You already know that in any right-angled triangle, the side opposite the right angle is called the hypotenuse. The other two sides of the triangle are called the opposite side and the adjacent side, depending on the acute angle with which we are working. The side opposite the acute angle is the opposite side and the side next to the acute angle is the adjacent side, as shown in the two diagrams below.



The three **trigonometric ratios** are called the sine ratio, the cosine ratio and the tangent ratio. We define them as follows:

$$\sin \theta = \frac{\text{opposite side}}{\text{hypotenuse}} \quad \cos \theta = \frac{\text{adjacent side}}{\text{hypotenuse}} \quad \tan \theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

We use the abbreviations sin for sine, cos for cosine and tan for tangent. So, in short, we have:

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} \quad \cos \theta = \frac{\text{adj}}{\text{hyp}} \quad \tan \theta = \frac{\text{opp}}{\text{adj}}$$

You can use the mnemonic SOH - CAH - TOA to memorise these ratios.

New word

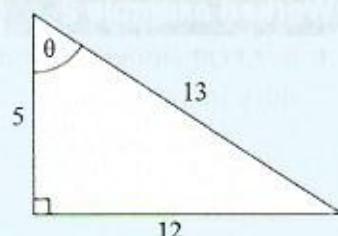
trigonometric ratios: the sine, cosine and tangent ratios, abbreviated to sin, cos and tan

Worked example 1

Given $\triangle PQR$ in the diagram alongside, find the values of $\sin \theta$, $\cos \theta$ and $\tan \theta$.

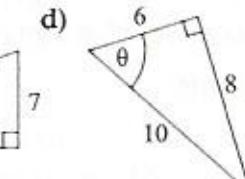
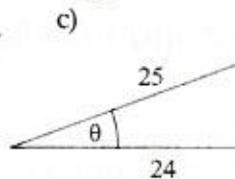
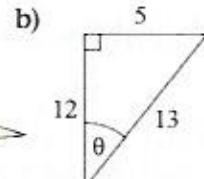
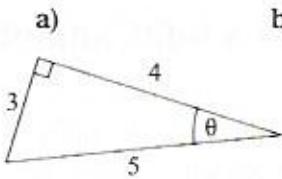
Answers

The ratios are as follows: $\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{12}{13}$
 $\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{5}{13}$
 $\tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{12}{5}$

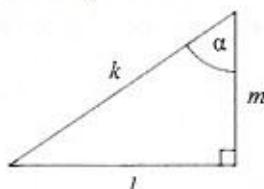
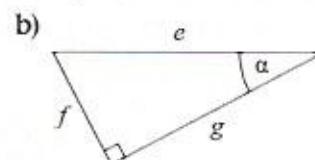
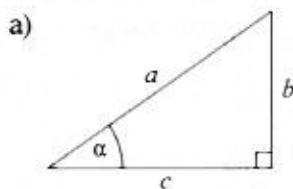


Activity 1

1 Write down the values of $\sin \theta$, $\cos \theta$ and $\tan \theta$ for each triangle below.



2 Write down the ratios for $\sin \alpha$, $\cos \alpha$ and $\tan \alpha$ for each triangle below.



Did you know?

We often use the Greek letters α (alpha), β (beta) and θ (theta) to denote angles in trigonometry. This is because the ancient Greeks contributed significantly to the development of trigonometry.

Calculate sides and angles of a right-angled triangle

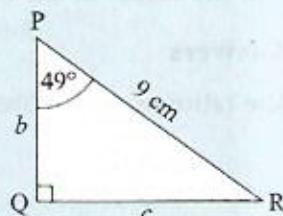
You can use trigonometric ratios to solve right-angled triangles. You can use a scientific calculator or trigonometric tables to do this.

Note

To solve a triangle means to calculate all the missing angles and sides of the triangle. Be aware of the following very important and useful fact: in any triangle, the longest side is always opposite the biggest angle and the shortest side is always opposite the smallest angle. Use this fact to check your answers when you have solved a triangle.

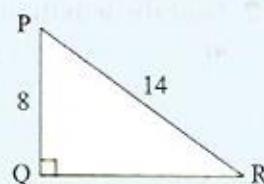
Worked example 2

1 In $\triangle PQR$ alongside, find all the unknown sides and angles.

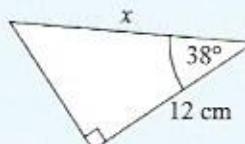


Worked example 2 (continued)

2 Calculate the size of $\angle R$ in the diagram alongside.



3 Find x in the diagram alongside.



Answers

$$1 \cos \angle P = \frac{\text{adj}}{\text{hyp}} = \frac{PQ}{PR}$$

$$\therefore \cos 49^\circ = \frac{b}{9}$$

$$\therefore b = 9 \times \cos 49^\circ = 5.90 \text{ cm} \quad (\text{rounded off to two decimal places})$$

$$\sin \angle P = \frac{\text{opp}}{\text{hyp}} = \frac{PQ}{PR}$$

$$\therefore \sin 49^\circ = \frac{c}{9}$$

$$\therefore c = 9 \times \sin 49^\circ = 6.79 \text{ cm} \quad (\text{rounded off to two decimal places})$$

$$\angle R = 180^\circ - 90^\circ - 49^\circ = 41^\circ \quad (\text{sum of angles of a triangle})$$

$$2 \sin \angle R = \frac{\text{opp}}{\text{hyp}} = \frac{8}{14}$$

$$\therefore \angle R = \sin^{-1} \frac{8}{14} = 34.8^\circ \quad (\text{rounded off to one decimal place})$$

$$3 \cos 38^\circ = \frac{12}{x}$$

$$\therefore x = \frac{12}{\cos 38^\circ} = 15.23 \text{ cm} \quad (\text{rounded off to two decimal places})$$

Note

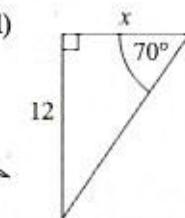
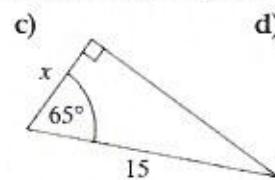
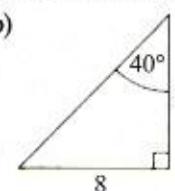
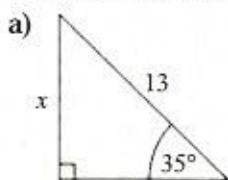
To calculate an angle if a ratio is given, we use the inverse ratio on the calculator. So, to find $\angle R$ if $\sin \angle R$ is given, we use the \sin^{-1} function, as you will see in the answer to Question 2.

Note

Unless specified otherwise, we round off the lengths of sides to two decimal places, and sizes of angles to one decimal place.

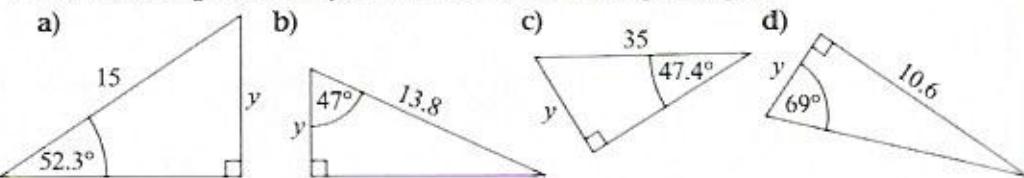
Activity 2

1 Calculate the value of x in each of the following triangles.

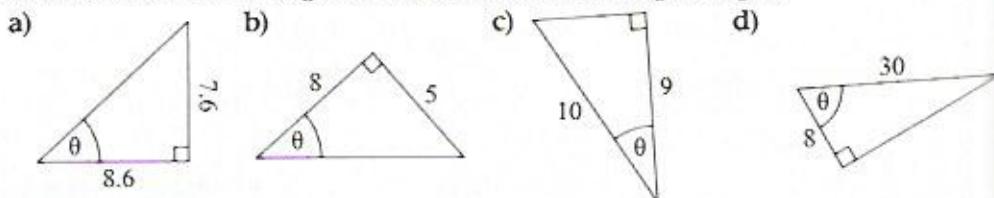


Activity 2 (continued)

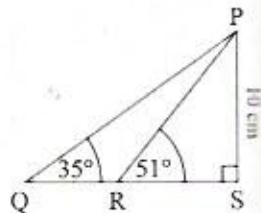
2 Find the length of side y in each of the following triangles.



3 Calculate the size of angle θ in each of the following triangles.

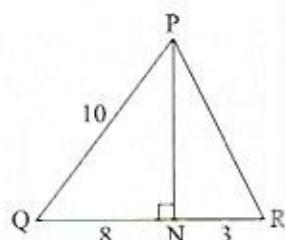


4 In $\triangle PQS$ alongside, $PS = 10 \text{ cm}$, $\angle PQR = 35^\circ$ and $\angle PRS = 51^\circ$. Calculate the length of QR .



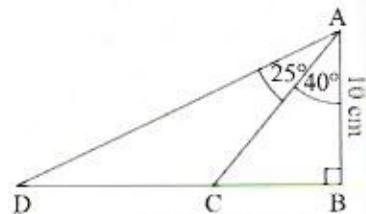
5 In $\triangle PQR$ alongside, $PQ = 10$, $QN = 8$ and $NR = 3$.

- Calculate the length of PN .
- Calculate the size of $\angle PRN$.

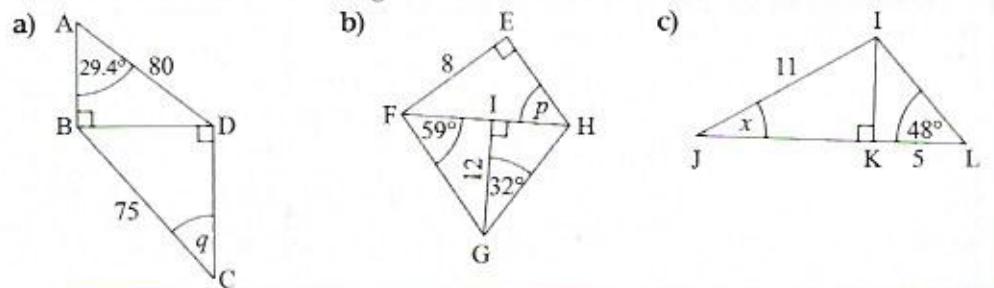


6 In $\triangle ABD$, $AB = 10 \text{ cm}$, $\angle DAC = 25^\circ$ and $\angle CAB = 40^\circ$. Calculate the lengths of:

- BD
- BC
- DC



7 Calculate the size of each angle marked with a letter.



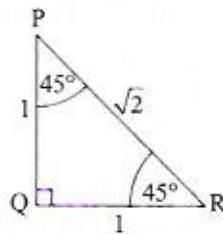
Work with special angles (30° , 45° , 60°)

The 45° triangle

In the diagram alongside, $\triangle PQR$ is a right-angled triangle with two acute angles of 45° .

$$\begin{aligned} \text{If } PQ = 1 \text{ unit then } QR = 1 \text{ unit} & \quad (\text{isosceles triangle}) \\ \therefore PR^2 &= 1^2 + 1^2 & & (\text{Theorem of Pythagoras}) \\ &= 2 \\ \therefore PR &= \sqrt{2} \text{ units} \\ \therefore \tan 45^\circ &= \frac{1}{1} = 1, \sin 45^\circ = \frac{1}{\sqrt{2}} \text{ and } \cos 45^\circ = \frac{1}{\sqrt{2}} \end{aligned}$$

So, any triangle with angles of 45° , 45° and 90° has sides in the ratio $1 : 1 : \sqrt{2}$.



The $30^\circ/60^\circ$ triangle

In the diagram alongside, $\triangle PQR$ is an equilateral triangle of sides 2 unit in length.

PS is the altitude of $\triangle PQR$ and $QS = SR = 1$ unit.

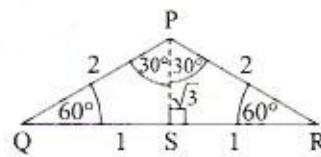
$$\begin{aligned} \text{In } \triangle PQS: \\ PS^2 &= PQ^2 - QS^2 & & (\text{Theorem of Pythagoras}) \\ &= 2^2 - 1^2 \\ &= 3 \\ \therefore PS &= \sqrt{3} \text{ units} \end{aligned}$$

Since $\triangle PQR$ is equilateral, $\angle B = \angle C = \angle A = 60^\circ$.

$$\therefore \cos 60^\circ = \frac{1}{2}, \sin 60^\circ = \frac{\sqrt{3}}{2} \text{ and } \tan 60^\circ = \frac{\sqrt{3}}{1} = \sqrt{3}$$

Note that $\angle QPS = 30^\circ$, so $\cos 30^\circ = \frac{\sqrt{3}}{2}$, $\sin 30^\circ = \frac{1}{2}$ and $\tan 30^\circ = \frac{1}{\sqrt{3}}$

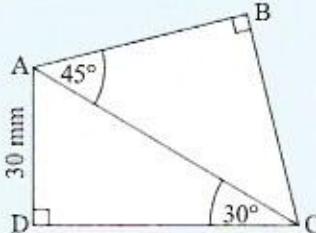
So, any triangle with angles of 30° , 60° and 90° has sides in the ratio $1 : \sqrt{3} : 2$.



Worked example 3

In the diagram alongside, $AD = 30$ mm.

Without using a calculator, calculate the length of AB. Leave your answer in surd form.



Answer

In $\triangle ACD$:

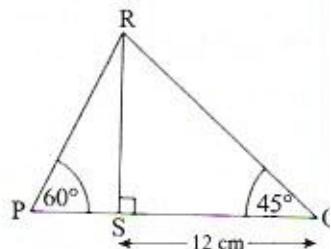
$$\begin{aligned} \sin 30^\circ &= \frac{30}{AC} \\ \therefore AC &= \frac{30}{\sin 30^\circ} = \frac{30}{\frac{1}{2}} = 60 \text{ mm} \end{aligned}$$

In $\triangle ABC$:

$$\begin{aligned} \cos 45^\circ &= \frac{AB}{60} \\ \therefore AB &= 60 \times \cos 45^\circ = 60 \times \frac{1}{\sqrt{2}} = \frac{60}{\sqrt{2}} \text{ mm} \end{aligned}$$

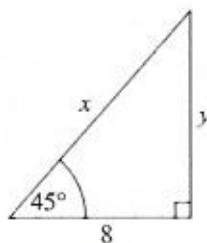
Activity 3

1 In the diagram alongside, $SQ = 12 \text{ cm}$.
Without using a calculator, calculate the length of PR . Leave your answer in surd form.

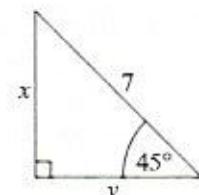


2 In each of the following, calculate the lengths of the sides marked x and y without using a calculator. Leave your answers in surd form, where necessary.

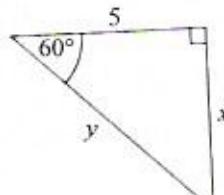
a)



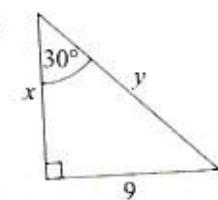
b)



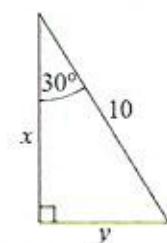
c)



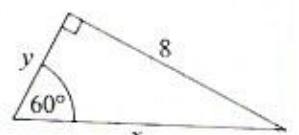
d)



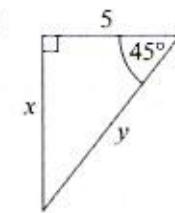
e)



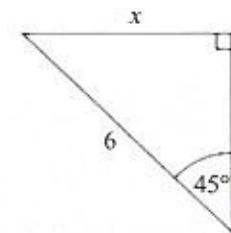
f)



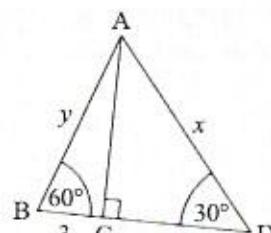
g)



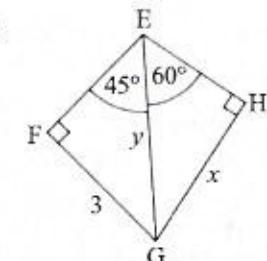
h)



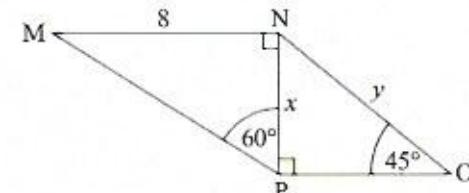
i)



j)



k)



Sine and cosine rules

Find sides and angles of non right-angled triangles

Thus far in this topic, you have worked with right-angled triangles only. In this sub-topic, you will discover how to solve triangles that do not have right angles. You will learn about two very useful rules for solving non right-angled triangles: the sine rule and the cosine rule.

The sine rule

In any $\triangle ABC$, the angles are denoted by capital letters A, B, C respectively, and the sides opposite these angles are denoted by a , b and c respectively.

Given: any acute-angled $\triangle ABC$

Prove: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Construction: Draw a perpendicular CD from C to AB, with D on AB.

Proof: Let CD be h .

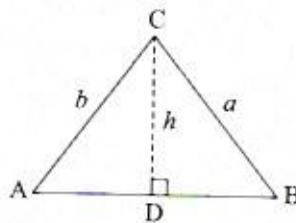
$$\therefore \sin A = \frac{h}{b} \text{ and } \sin B = \frac{h}{a}$$

$$\therefore h = b \sin A = a \sin B$$

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B}$$

Similarly, we can prove that $\frac{a}{\sin A} = \frac{c}{\sin C}$.

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



It is also possible to prove the sine rule for any obtuse-angled $\triangle ABC$.

So, for any $\triangle ABC$, the sine rule states that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

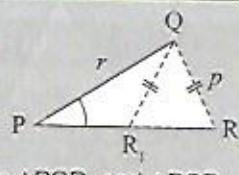
An alternative form of the sine rule is as follows: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

We use the sine rule to solve non right-angled triangles, in which:

- two angles and any side are given
- two sides and the non-included angle are given.

Note

The sine rule can sometimes have two solutions for a given triangle. We call this the **ambiguous case**. The ambiguous case can occur if we are given two sides and a non-included angle, where the non-included angle is opposite the shorter of the two given sides. An example of this is shown alongside. If we are given $\angle P$, p and r , where p is shorter than r , then there are two possible solutions to this triangle: $\triangle PQR_1$ and $\triangle PQR_2$.



New words

sine rule: a rule to solve non right-angled triangles, when two angles and any side are given, or two sides and the non-included angle are given

ambiguous case: this occurs when the sine rule has two solutions for a given triangle

Worked example 4

- In $\triangle PQR$, $p = 12.5$ cm, $r = 16.7$ and $\angle R = 112^\circ$. Solve $\triangle PQR$.
- In $\triangle ABC$, $\angle A = 63^\circ$, $b = 18$ and $a = 17$. Solve $\triangle ABC$.

Answers

- If no sketch is given, it is a good idea to draw your own rough sketch showing the given information.

Since two sides and the non-included angle are given, we will use the sine rule.

$$\frac{\sin P}{p} = \frac{\sin R}{r} \quad (\text{sine rule})$$

$$\therefore \frac{\sin P}{12.5} = \frac{\sin 112^\circ}{16.7}$$

$$\therefore \sin P = \frac{12.5 \times \sin 112^\circ}{16.7}$$

$$\therefore \angle P = \sin^{-1} \left(\frac{12.5 \times \sin 112^\circ}{16.7} \right) = 43.9^\circ$$

$$\therefore \angle Q = 180^\circ - (43.9^\circ + 112^\circ) = 24.1^\circ$$

$$\frac{q}{\sin Q} = \frac{r}{\sin R}$$

$$\therefore \frac{q}{\sin 24.1^\circ} = \frac{16.7}{\sin 112^\circ}$$

$$\therefore q = \frac{(16.7 \times \sin 24.1^\circ)}{\sin 112^\circ} = 7.35 \text{ cm}$$

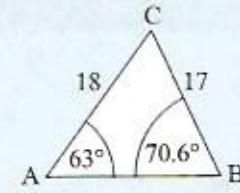
- $\frac{\sin B}{18} = \frac{\sin 63^\circ}{17} \quad (\text{sine rule})$

$$\therefore \sin B = \frac{(18 \times \sin 63^\circ)}{17}$$

$$\therefore \angle B = \sin^{-1} \left(\frac{(18 \times \sin 63^\circ)}{17} \right) = 70.6^\circ$$

There are two possible solutions for $\angle B$. The second solution is $\angle B = 180^\circ - 70.6^\circ = 109.4^\circ$. This is an example of the ambiguous case.

Solution 1:

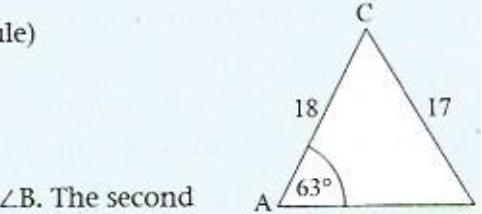
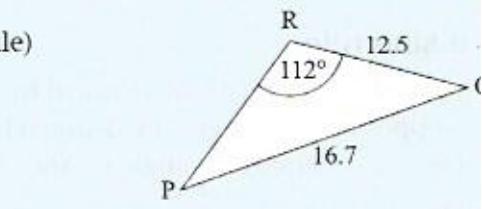


$$\angle C = 180^\circ - (63^\circ + 70.6^\circ) = 46.4^\circ$$

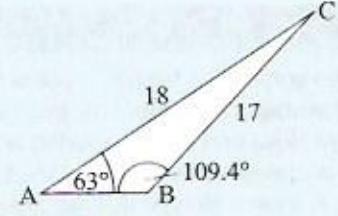
$$\frac{c}{\sin 46.4^\circ} = \frac{17}{\sin 63^\circ}$$

$$\therefore c = \frac{(17 \times \sin 46.4^\circ)}{17} = 13.82$$

So: $\angle B = 70.6^\circ$, $\angle C = 46.4^\circ$
and $c = 13.82$



Solution 2:



$$\angle C = 180^\circ - (63^\circ + 109.4^\circ) = 7.6^\circ$$

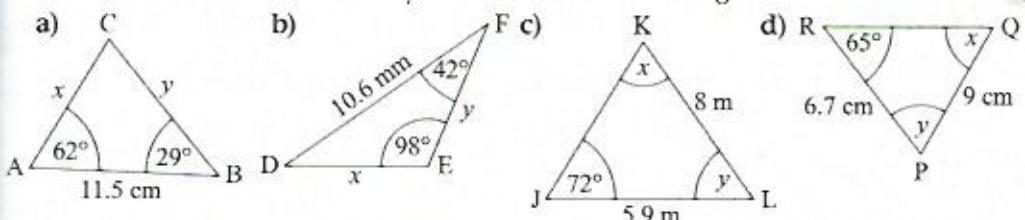
$$\frac{c}{\sin 7.6^\circ} = \frac{17}{\sin 63^\circ}$$

$$\therefore c = \frac{(17 \times \sin 7.6^\circ)}{\sin 63^\circ} = 2.52$$

or $\angle B = 109.4^\circ$, $\angle C = 7.6^\circ$ and $c = 2.52$

Activity 4

1 Calculate the values of x and y in each of the following.



2 Solve each of the following triangles.

- In $\triangle ABC$, $\angle B = 95^\circ$, $\angle C = 30^\circ$ and $c = 18.9$ cm.
- In $\triangle DEF$, $\angle D = 125.4^\circ$, $d = 83$ m and $e = 27.4$ m.
- In $\triangle HJK$, $\angle J = 38^\circ$, $\angle K = 76.2^\circ$ and $j = 33$ cm.
- In $\triangle ABC$, $a = 4.8$ mm, $b = 5.2$ mm and $\angle A = 48^\circ$.
- In $\triangle PQR$, $p = 6$ cm, $\angle Q = 69^\circ$ and $q = 12.6$ cm.

The cosine rule

Given: any acute-angled $\triangle ABC$

Prove: $a^2 = b^2 + c^2 - 2bc \cos A$

Construction: Draw a perpendicular CD from C to AB, with D on AB.

Proof: Let CD be h .

In $\triangle ACD$:

$$h^2 + AD^2 = b^2 \quad \textcircled{1} \quad (\text{Theorem of Pythagoras})$$

$$\text{and } AD = b \cos A \quad \textcircled{2}$$

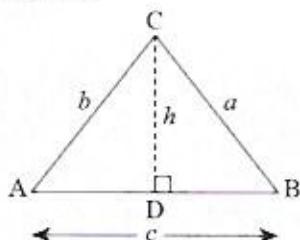
In $\triangle BCD$:

$$a^2 = h^2 + BD^2 \quad (\text{Theorem of Pythagoras})$$

$$\begin{aligned} \therefore a^2 &= h^2 + (c - AD)^2 \\ &= h^2 + c^2 - 2 \times c \times AD + AD^2 \\ &= (h^2 + AD^2) + c^2 - 2 \times c \times AD \quad \textcircled{3} \end{aligned}$$

Substitute $\textcircled{1}$ and $\textcircled{2}$ into $\textcircled{3}$:

$$\therefore a^2 = b^2 + c^2 - 2bc \cos A$$



It is also possible to prove the cosine rule for any obtuse-angled $\triangle ABC$.

So, for any $\triangle ABC$, the **cosine rule** states that

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

An alternative form of the cosine rule is as follows:

$$\cos A = \frac{(b^2 + c^2 - a^2)}{2bc}.$$

We use the cosine rule to solve non right-angled triangles, in which:

- two sides and the included angle are given
- three sides are given.

New word

cosine rule: a rule to solve non right-angled triangles, when two sides and the included angle are given, or three sides are given

Worked example 5

1 Solve $\triangle ABC$, if $a = 15$, $b = 25$ and $c = 63^\circ$.

2 Solve $\triangle ABC$, if $a = 13$, $b = 11$ and $c = 16$.

Answers

$$\begin{aligned} 1 \quad c^2 &= a^2 + b^2 - 2ab \cos C && \text{(cosine rule)} \\ &= 15^2 + 25^2 - 2(15)(25) \cos 63^\circ \\ \therefore c &= \sqrt{15^2 + 25^2 - 2(15)(25) \cos 63^\circ} = 22.57 \end{aligned}$$

To find the second angle, we will use the sine rule.

$$\begin{aligned} \frac{b}{\sin B} &= \frac{c}{\sin C} && \text{(sine rule)} \\ \frac{25}{\sin B} &= \frac{22.57}{\sin 63^\circ} \\ \sin B &= \frac{25 \times \sin 63^\circ}{22.57} \\ \therefore \angle B &= \sin^{-1} \left(\frac{25 \times \sin 63^\circ}{22.57} \right) = 80.7^\circ \end{aligned}$$

$\therefore \angle A = 180^\circ - (63^\circ + 80.7^\circ) = 36.3^\circ$ (sum of angles of a triangle)

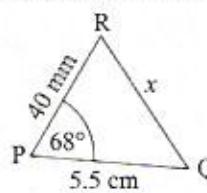
$$\begin{aligned} 2 \quad \cos C &= \frac{(a^2 + b^2 - c^2)}{2ab} && \text{(cosine rule)} \\ \therefore \angle C &= \cos^{-1} \left(\frac{(13^2 + 11^2 - 16^2)}{2(13)(11)} \right) = 83.17^\circ \\ \frac{b}{\sin B} &= \frac{c}{\sin C} && \text{(sine rule)} \\ \frac{11}{\sin B} &= \frac{16}{\sin 83.17^\circ} \\ \sin B &= \frac{11 \times \sin 83.17^\circ}{16} \\ \therefore \angle B &= \sin^{-1} \left(\frac{11 \times \sin 83.17^\circ}{16} \right) = 43.0^\circ \end{aligned}$$

$\therefore \angle A = 180^\circ - (83.17^\circ + 43^\circ) = 53.8^\circ$ (sum of angles of a triangle)

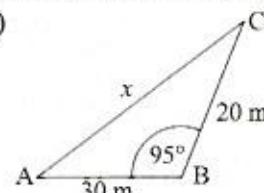
Activity 5

1 Calculate the value of x in each of the following.

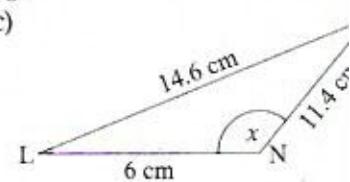
a)



b)



c)



2 Solve each of the following triangles.

a)

In $\triangle ABC$, $\angle A = 60^\circ$, $b = 10$ cm and $c = 7.5$ cm.

b)

In $\triangle PQR$, $\angle Q = 130^\circ$, $r = 6.2$ m and $p = 5$ m.

c)

In $\triangle ABC$, $\angle C = 25^\circ$, $b = 3.3$ cm and $a = 6$ cm.

d)

In $\triangle LMN$, $l = 45$ mm, $m = 30$ mm and $n = 65$ mm.

e)

In $\triangle PQR$, $p = 14$ m, $q = 11.4$ m and $r = 9$ m.

Calculate the area of non right-angled triangles

The area of any triangle can be expressed in terms of two sides and the included angle of the triangle.

Given: any acute-angled $\triangle ABC$

Prove: Area of $\triangle ABC = \frac{1}{2} ab \sin C$

Construction: Draw a perpendicular AD from A to BC, with D on BC.

Proof: Let AD be h .

In $\triangle ACD$:

$$\sin C = \frac{h}{b}$$

$$\therefore h = b \sin C$$

①

$$\text{Area of } \triangle ABC = \frac{1}{2} \times \text{base} \times \perp \text{height}$$

$$= \frac{1}{2} \times a \times h$$

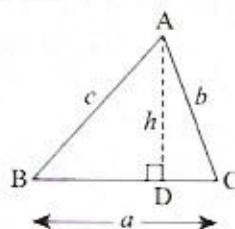
②

Substitute ① into ②:

$$\text{Area of } \triangle ABC = \frac{1}{2} ab \sin C$$

It is also possible to prove the area rule for any obtuse-angled $\triangle ABC$.

So, for any $\triangle ABC$, the **area rule** states that the area of $\triangle ABC = \frac{1}{2} ab \sin C$.



New word

area rule: the rule to calculate the area of any triangle, when two sides and the included angle are given

Worked example 6

Find the area of $\triangle ABC$ if $a = 7.5$ cm, $b = 11$ cm and $C = 100^\circ$.

Answer

$$\begin{aligned} \text{Area} &= \frac{1}{2} ab \sin C \\ &= \frac{1}{2} \times 7.5 \times 11 \times \sin 100^\circ = 40.62 \text{ cm}^2 \end{aligned}$$

Activity 6

1 In each of the following, find the area of $\triangle ABC$.

(Hint: In Questions d) and e), first use the cosine rule to calculate the size of any angle. Then use the area rule to find the area of $\triangle ABC$.)

a) $\angle A = 43^\circ$, $b = 16$ cm and $c = 12$ cm

b) $a = 6$ cm, $\angle B = 52^\circ$ and $c = 4$ cm

c) $a = 3$ m, $b = 4$ m and $\angle C = 120^\circ$

d) $a = 4$ cm, $b = 6$ cm and $c = 8$ cm

e) $a = 17$ mm, $b = 13$ mm and $c = 19$ mm

2 $\triangle ABC$ has an area of 40 cm^2 . If $b = 12$ cm and $c = 20$ cm, calculate the size of $\angle A$.

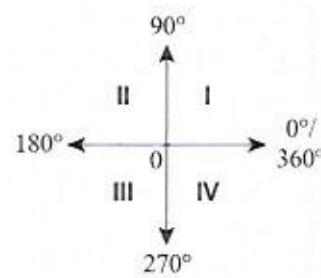
In this sub-topic, you will work with trigonometry on the Cartesian plane.

Determine the signs of the three trigonometric ratios in the quadrants

You already know that the Cartesian plane is divided into four quadrants, as shown in the diagram alongside.

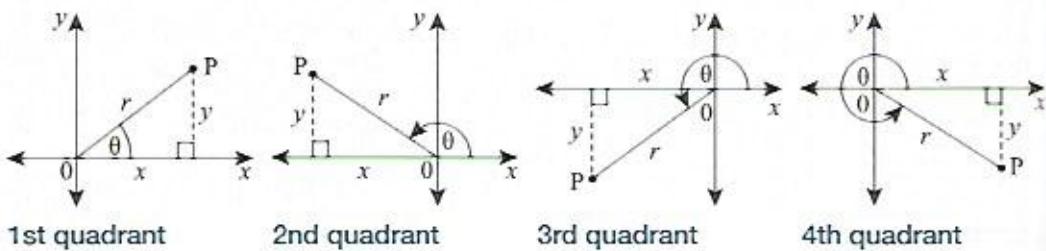
Note the following:

- the quadrants are numbered from I to IV, in an anti-clockwise direction
- angles between 0° and 90° fall in the 1st quadrant
- angles between 90° and 180° fall in the 2nd quadrant
- angles between 180° and 270° fall in the 3rd quadrant
- angles between 270° and 360° fall in the 4th quadrant
- in the 1st quadrant, $x > 0$ and $y > 0$
- in the 2nd quadrant, $x < 0$ and $y > 0$
- in the 3rd quadrant, $x < 0$ and $y < 0$
- in the 4th quadrant, $x > 0$ and $y < 0$.



The CAST diagram

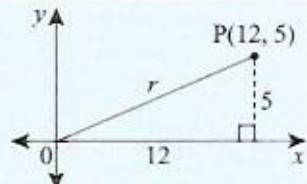
In the diagrams below, $P(x, y)$ is a point in each of the four quadrants of the Cartesian plane. The length of $OP = r$, where $r > 0$ and $\angle POX = \theta$.



In all four diagrams, we have: $\sin \theta = \frac{y}{r}$, $\cos \theta = \frac{x}{r}$, $\tan \theta = \frac{y}{x}$. The signs of x and y differ from quadrant to quadrant, so the values of \sin , \cos , and \tan will differ as well.

Worked example 7

- In the diagram alongside, P has coordinates $(12, 5)$. Find the values of $\sin \angle XOP$, $\cos \angle XOP$ and $\tan \angle XOP$.
- Find the value of $\sin \theta$ and $\tan \theta$ if $\cos \theta = -\frac{2}{3}$ and $180^\circ < \theta < 270^\circ$.



Worked example 7 (continued)

Answers

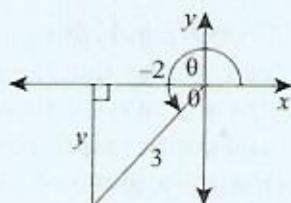
$$\begin{aligned}
 1 \quad r &= \sqrt{x^2 + y^2} && \text{(Theorem of Pythagoras)} \\
 &= \sqrt{12^2 + 5^2} \\
 &= \sqrt{169} \\
 &= 13
 \end{aligned}$$

$$\begin{aligned}
 \sin \angle XOP &= \frac{y}{r} = \frac{5}{13} \\
 \cos \angle XOP &= \frac{x}{r} = \frac{12}{13} \\
 \tan \angle XOP &= \frac{y}{x} = \frac{5}{12}
 \end{aligned}$$

2 $180^\circ < \theta < 270^\circ$, so θ falls in the 3rd quadrant.

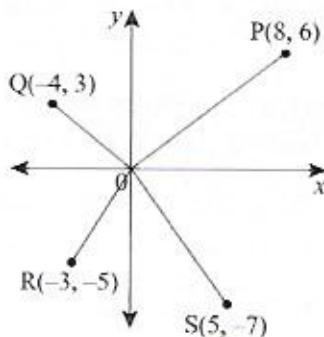
Draw a diagram of this situation.

$$\begin{aligned}
 y &= -\sqrt{r^2 - x^2} && \text{(Theorem of Pythagoras)} \\
 &= -\sqrt{3^2 - (-2)^2} \\
 &= -\sqrt{9 - 4} \\
 &= -\sqrt{5} && \text{(Note that } y \text{ is negative in the 3rd quadrant)} \\
 \therefore \sin \theta &= \frac{y}{r} = -\frac{\sqrt{5}}{3} \text{ and } \tan \theta = -\frac{\sqrt{5}}{2} = \frac{\sqrt{5}}{2}
 \end{aligned}$$



Activity 7

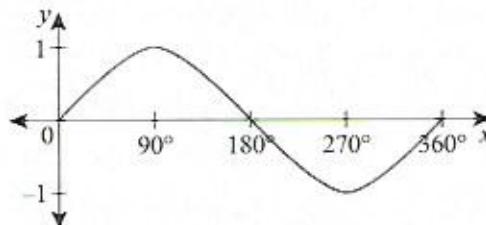
- For each of the following points, calculate the values of $\sin \theta$, $\cos \theta$ and $\tan \theta$ with the aid of a diagram, if θ is the angle between the line OP and the positive x -axis.
 - $P(6, 8)$
 - $P(-4, 3)$
 - $P(15, -8)$
 - $P(-5, -12)$
 - $P(24, 7)$
 - $P(4, -3)$
- P , Q , R and S are points on the Cartesian plane, as shown in the diagram alongside.
 - Find the lengths of OP , OQ , OR and OS .
 - Write down the value of $\sin \theta$, $\cos \theta$ and $\tan \theta$ for each of OP , OQ , OR and OS , where θ is the angle between the line and the positive x -axis.
- Find the values of $\sin \theta$ and $\cos \theta$ given that $\tan \theta = -\frac{5}{13}$ and θ is in the 4th quadrant.
- If $\sin \theta = \frac{3}{5}$ and θ is in the 2nd quadrant, find the values of $\cos \theta$ and $\tan \theta$.



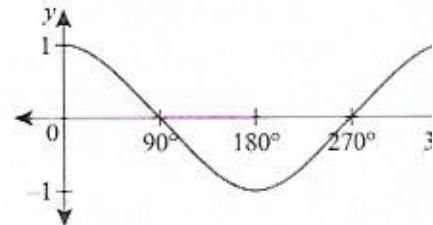
Draw graphs for sine, cosine and tangent curves

Below are the graphs of $\sin x$ and $\cos x$ for $x \in [0^\circ, 360^\circ]$.

The sine graph:



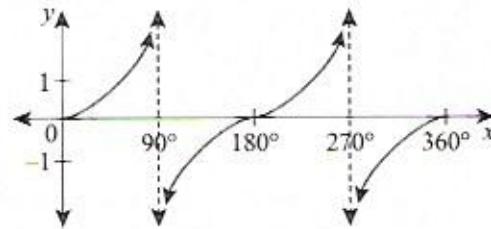
The cosine graph:



Note the following points about the sine and cosine graphs:

- The sine and cosine graphs have a wave shape.
- The sine and cosine graphs have a minimum value of -1 and a maximum value of 1 . We say that the sine and cosine graphs have an **amplitude** of 1 .
- The sine and cosine graphs complete a full cycle in 360° . We say that the sine and cosine graphs have a **period** of 360° .

Below is the graph of $\tan x$ for $x \in [0^\circ, 360^\circ]$.



Note the following points about the tangent graph:

- The tangent graph does not have a minimum value or a maximum value.
- The tangent graph is undefined where $x = 90^\circ$ and 270° . We say that the tangent graph has **asymptotes** at 90° and at 270° .
- The tangent graph completes a full cycle in 180° . So, the tangent graph has a period of 180° .

New words

amplitude: half the distance between the highest and lowest points of a wave

period: the distance on the horizontal axis of one full cycle of a graph

asymptote: a straight line to which a graph draws closer and closer, without ever touching it

Activity 8

- 1 Use the sine, cosine and tangent graphs above to find the values of:
a) $\sin 0^\circ$ b) $\sin 90^\circ$ c) $\sin 180^\circ$ d) $\sin 360^\circ$
e) $\cos 0^\circ$ f) $\cos 180^\circ$ g) $\cos 270^\circ$ h) $\cos 360^\circ$
- 2 Use the sine, cosine and tangent graphs above to find the value(s) of θ for which:
a) $\sin \theta = 1$ b) $\sin \theta = 0$ c) $\sin \theta = -1$ d) $\cos \theta = 1$
e) $\cos \theta = 0$ f) $\cos \theta = -1$ g) $\tan \theta = 0$

Solve trigonometric equations

As you have already seen, $\sin \theta$, $\cos \theta$ and $\tan \theta$ can be positive or negative, depending on the quadrant within which θ falls. We can summarise this in the CAST diagram alongside.

We use the CAST diagram to determine the signs of the \sin , \cos and \tan ratios in the different quadrants of the Cartesian plane.

In the 1st quadrant, All are positive.

In the 2nd quadrant, Sin is positive. Cos and tan are negative.

In the 3rd quadrant, Tan is positive. Sin and cos are negative.

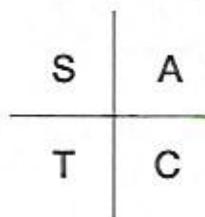
In the 4th quadrant, Cos is positive. Sin and tan are negative.

The CAST diagram is very useful when we solve trigonometric equations. A trigonometric equation will usually have two solutions.

To solve equations of the form $\sin \theta = k$,

$\cos \theta = k$ or $\tan \theta = k$:

- Find the **reference angle**, α , in the 1st quadrant.
- Determine in which two quadrants θ will lie.
- Find the corresponding angles in the two quadrants.



New words

CAST diagram: a visual aid used to determine the signs of the \sin , \cos and \tan ratios in the different quadrants of the Cartesian plane

reference angle: an angle in the first quadrant of the Cartesian plane, used to determine the solution of a trigonometric equation

Worked example 8

Solve the following equations for $0^\circ \leq \theta \leq 360^\circ$.

1 $\sin \theta = 0.866$

2 $\cos \theta = -0.35$

Answers

1 In the 1st quadrant,

$$\alpha = \sin^{-1} 0.866 = 60^\circ$$

Sin is positive in the 1st and 2nd quadrants.

In the 1st quadrant, $\theta = 60^\circ$

In the 2nd quadrant,

$$\theta = 180^\circ - \alpha = 180^\circ - 60^\circ = 120^\circ$$

$$\therefore \theta = 60^\circ \text{ or } \theta = 120^\circ$$

2 In the 1st quadrant

$$\alpha = \cos^{-1} 0.35 = 69.5^\circ$$

Cos is negative in the 2nd and 3rd quadrants.

In the 2nd quadrant,

$$\theta = 180^\circ - \alpha = 180^\circ - 69.5^\circ = 110.5^\circ$$

In the 3rd quadrant,

$$\theta = 180^\circ + \alpha = 180^\circ + 69.5^\circ = 249.5^\circ$$

$$\therefore \theta = 110.5^\circ \text{ or } \theta = 249.5^\circ$$

Activity 9

Solve each of the following equations for $0^\circ \leq \theta \leq 360^\circ$.

1 $\sin \theta = 0.64$	2 $\cos \theta = 0.42$	3 $\tan \theta = 3.73$	4 $\sin \theta = -0.93$
5 $\tan \theta = -0.81$	6 $\sin \theta = -0.5$	7 $\tan \theta = 1.75$	8 $\cos \theta = -0.98$
9 $\sin \theta = -0.56$	10 $\cos \theta = 0.34$	11 $\tan \theta = -5.67$	12 $\sin \theta = 0.3$

Use trigonometry to solve practical problems

There are many useful applications of trigonometry in real life. In this sub-topic, you will solve practical problems in two dimensions, including problems that involve compass bearings. You will also solve practical problems in three dimensions.

Solve practical problems in two dimensions

Worked example 9

A farm is in the shape of an irregular quadrilateral ABCD. $AB = 360$ m, $BC = 240$ m, $CD = 200$ m, $AD = 64$ m and $\angle DAB = 89^\circ$. Calculate the area of the farm.

Answer

First draw a diagram of the quadrilateral.

Join points B and D to form two triangles.

In $\triangle ABD$:

$$a^2 = b^2 + d^2 - 2bd \cos A \quad (\text{cosine rule})$$

$$\therefore BD = \sqrt{64^2 + 360^2 - 2(64)(360)\cos 89^\circ} = 364.54 \text{ m}$$

In $\triangle BCD$:

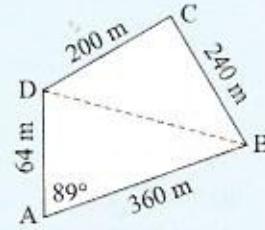
$$\cos C = \frac{(b^2 + d^2 - c^2)}{2bd} \quad (\text{cosine rule})$$

$$\therefore \angle C = \cos^{-1} \frac{(200^2 + 240^2 - 364.54^2)}{2(200)(240)} = 111.6^\circ$$

$$\text{Area of } \triangle ABD = \frac{1}{2} bd \sin A = \frac{1}{2} \times 64 \times 360 \times \sin 89^\circ = 11\ 518.25 \text{ m}^2$$

$$\text{Area of } \triangle BCD = \frac{1}{2} bd \sin C = \frac{1}{2} \times 64 \times 360 \times \sin 111.6^\circ = 10\ 711.03 \text{ m}^2$$

$$\therefore \text{Area of } ABCD = 11\ 518.25 \text{ m}^2 + 10\ 711.03 \text{ m}^2 = 22\ 229.28 \text{ m}^2$$



Activity 10

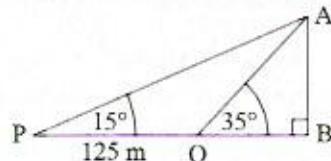
- 1 A ladder 15 m long is set against a wall and it makes an angle of 82° with the ground.
 - a) How far up the wall does the ladder reach?
 - b) How far is the foot of the ladder from the wall?
- 2 An electric pole 12 m high is supported by a wire 15 m long fixed to the top of the pole and to the ground.
 - a) Calculate the angle between the wire and the ground.
 - b) Calculate the distance on the ground between the foot of the pole and the wire.

Activity 10 (continued)

3 A land surveyor needs to find the distance between P and Q which are on opposite sides of the banks of a river. He walks along the river bank from point Q and finds another point R such that $\angle PQR = 112^\circ$ and $\angle PRQ = 27^\circ$. The distance between Q and R is 242 m. Calculate the distance between P and Q.

4 A circle with centre O has a radius of 6.3 cm. AB is a chord on the circle such that $\angle AOB = 150^\circ$. Calculate the area of $\triangle ABO$.

5 In the diagram alongside, points P and Q are in a direct line with the foot of a vertical cliff, AB. From point P, the angle of elevation of the top of the cliff is 15° . From point Q, the angle of elevation is 35° . The distance from P to Q is 125 m. Calculate the height of the cliff.



Solve practical problems involving bearings

You have already worked with bearings in Grade 10. Remember that bearings are measured from north in a clockwise direction and are always written with three digits. So, for example, we write a bearing of east as 090° , and not as 90° .

Worked example 10

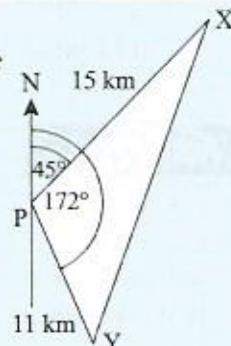
Village X is 15 km on a bearing of 045° from a crossroad marked P. Village Y is 11 km on a bearing of 172° from the same crossroad. Calculate the distance between the two villages.

Answer

$$\angle XPY = 172^\circ - 45^\circ = 127^\circ$$

$$a^2 = b^2 + c^2 - 2ac \cos A$$

$$\therefore XY = \sqrt{(11^2 + 15^2 - 2 \times 11 \times 15 \times \cos 127^\circ)} = 23.34 \text{ km}$$



Activity 11

Work in pairs for this activity. For each question, draw a diagram to help you.

- 1 A girl walked 1 400 m due east from X to Y. She then walked 700 m due north from Y to Z.
 - Calculate the bearing of Z from X.
 - Calculate the length of XZ.
- 2 Village B is 12 km on a bearing of 030° from village A. Village C is 7.5 km on a bearing of 155° from village A. Calculate the distance between village B and village C.
- 3 An aeroplane flies 250 km on a bearing of 072° , and then flies 190 km on a bearing of 010° .
 - How far east is the aeroplane from its starting point?
 - How far north is the aeroplane from its starting point?

Solve practical problems in three dimensions

When solving a problem in three dimensions, look for cross-sections or faces that are two-dimensional and then apply your knowledge of two-dimensional trigonometry to solve the problem in stages.

Worked example 11

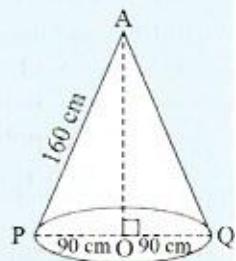
In the diagram alongside, a right circular cone has a slant height of 160 cm. The radius of its base is 90 cm.

- Calculate the angle which the slant height makes with the horizontal.
- Hence, calculate the perpendicular height of the cone.

Answers

a) $\cos \angle APO = \frac{OP}{AP} = \frac{90}{160}$
 $\therefore \angle APO = \cos^{-1}\left(\frac{90}{160}\right) = 55.8^\circ$

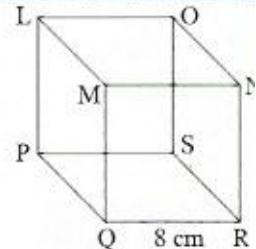
b) $\frac{AO}{OP} = \tan \angle APO$
 $\therefore \frac{AO}{90} = \tan 55.8^\circ$
 $\therefore AO = 90 \times \tan 55.8^\circ = 132.43 \text{ cm}$



Activity 12

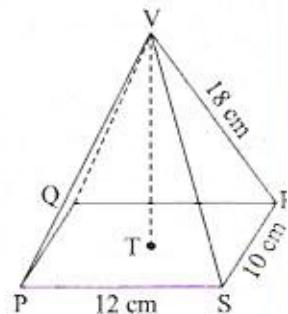
1 The cube in the diagram alongside has side lengths of 8 cm.

- Calculate the lengths of PR, QL and OQ.
- Calculate the sizes of $\angle QOS$ and $\angle QLP$.



2 The diagram alongside shows a rectangular pyramid with vertex V and edges VP, VQ, VR, and VS, each 18 cm long. The rectangular base has $PQ = RS = 10 \text{ cm}$ and $PS = QR = 12 \text{ cm}$.

- Calculate the length of QS.
- Calculate VT, the height of the pyramid.
- Calculate the size of the angle between
 - the base and VR,
 - the base and $\triangle VSR$,
 - the base and $\triangle VPS$.



Summary

Trigonometric ratios

- The three trigonometric ratios are the sine ratio, the cosine ratio and the tangent ratio: $\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$, $\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$ and $\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$.
- Any triangle with angles of 45° , 45° and 90° has sides in the ratio $1 : 1 : \sqrt{2}$.
- Any triangle with angles of 30° , 60° and 90° has sides in the ratio $1 : \sqrt{3} : 2$.

Sine and cosine rules

- The sine rule states that for any $\triangle ABC$, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ and $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$. We use the sine rule when two angles and any side are given, or two sides and the non-included angle are given.
- The cosine rule states that for any $\triangle ABC$, $a^2 = b^2 + c^2 - 2bc \cos A$ or $\cos A = \frac{(b^2 + c^2 - a^2)}{2bc}$. We use the cosine rule when two sides and included angle are given, or three sides are given.

Area of triangles

- The area rule states that for any $\triangle ABC$, the area of $\triangle ABC = \frac{1}{2}ab \sin C$.

Trigonometry on the Cartesian plane

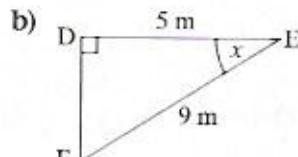
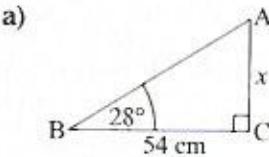
- The sine and cosine graphs have a minimum value of -1 , a maximum value of 1 , an amplitude of 1 and a period of 360° .
- The tangent graph has asymptotes at 90° and at 270° and a period of 180° .
- To solve equations of the form $\sin \theta = k$, $\cos \theta = k$ or $\tan \theta = k$, find the reference angle, α , in the 1st quadrant and determine in which two quadrants θ will lie.

Applications of trigonometry

- Bearings are measured clockwise from north and are written with 3 digits.
- When solving a problem in three dimensions, look for two-dimensional cross-sections or faces.

Revision exercises (remedial)

1 Calculate the value of x in each of the following triangles. $(2 \times 2 = 4)$



Revision exercises

2 Solve $\triangle PQR$, if $\angle Q = 124^\circ$, $r = 16 \text{ cm}$ and $q = 29 \text{ cm}$. (6)

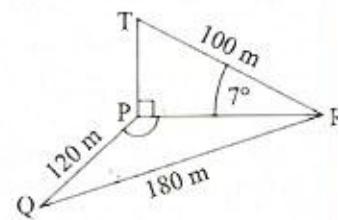
3 On the same set of axes, draw the graphs of $\sin x$, $\cos x$ and $\tan x$ for $x \in [0^\circ, 180^\circ]$. (6)

4 A man cycles 100 km on a bearing of 135° and then cycles for 112 km due west. He then cycles directly to his starting point.

- Draw a neat diagram of this situation. Show all the given information. (3)
- Calculate the total distance that he cycled. (3)

5 In the diagram alongside, $\triangle PQR$ represents a horizontal triangular field. TP is a tree in the corner of the field. $PQ = 120 \text{ m}$, $QR = 180 \text{ m}$, $TR = 100 \text{ m}$ and $\angle TRP = 7^\circ$.

- Calculate the height of the tree. (2)
- Calculate the size of $\angle QPR$. (4)
- Find the area of the field PQR. (2)



Total marks: 30

Assessment exercises

1 Solve the equation $\tan \theta = -5.85$ for $0^\circ \leq \theta \leq 360^\circ$. (2)

2 A ship sails from X to Y on a bearing of 110° for a distance of 40 km. It then sails for 50 km on a bearing of 170° from Y to Z.

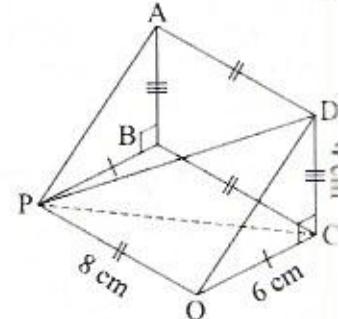
- Draw a neat diagram of this situation. Show all the given information. (3)
- Calculate the distance XZ. (2)
- Find the bearing of Z from X. (3)

3 A chord of length 14.4 cm subtends an angle of 68° at the centre of a circle. Find the radius of the circle. (3)

4 In the diagram alongside, PADQ and PBCQ are rectangles with $PQ = BC = AD = 8 \text{ cm}$ and $QC = PB = 6 \text{ cm}$. $CD = 4 \text{ cm}$. $ABCD \perp PBCQ$.

- Calculate the lengths of PC and PD. (5)
- Calculate the sizes of $\angle APB$, $\angle PDC$ and $\angle ADP$. (6)

5 PQRS is a kite. Diagonal PR bisects diagonal SQ at M. $PM = 5 \text{ cm}$, $MR = 9 \text{ cm}$ and $SQ = 12 \text{ cm}$. Calculate the sizes of the angles of the kite and the lengths of its sides. (6)



Total marks: 30

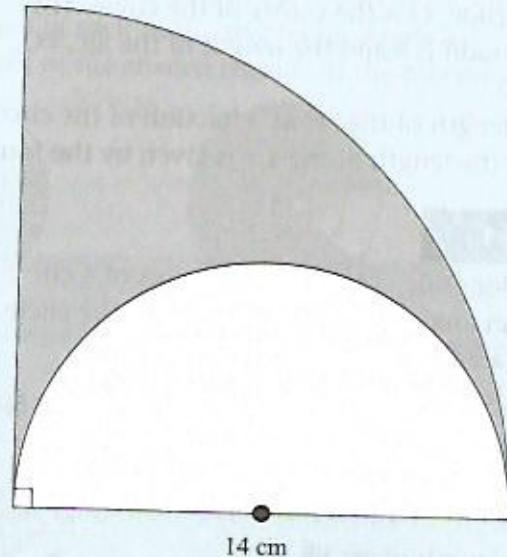


Sub-topics	Specific Outcomes
Area	<ul style="list-style-type: none">Calculate the area of a sectorCalculate the surface area of three-dimensional figures
Volume	<ul style="list-style-type: none">Calculate the volume of prismsSolve problems involving area and volume

Starter activity

Work in pairs for this activity.

- Calculate the area of a rectangle with length 30.2 cm and breadth 25.5 cm.
- A rectangle has length 25 cm and breadth 9 cm.
Find the side of a square of which the area is equal to the area of the rectangle.
- Calculate the area of the shaded part of the sector below.
The diameter of the semi-circle is 14 cm.



The **area** of a two-dimensional figure is the space that the figure occupies.

New word

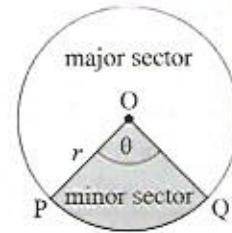
area: the amount of space that a two-dimensional figure occupies

Calculate the area of a sector

You already know that a sector is any part of a circle bounded by two radii.

In the diagram alongside, imagine that the circle is cut along radii OP and OQ . Two sectors are formed. The larger sector is the major sector and the smaller sector is the minor sector. Note that the area of a sector is proportional to the angle subtended at the centre of the circle.

We can express the area of the sector as a fraction of the area of the whole circle as $\frac{\theta}{360^\circ}$, so the area of the sector is given by the formula: $A = \frac{\theta}{360^\circ} \times \pi r^2$.

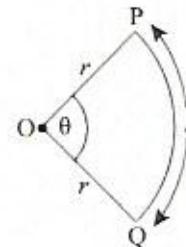


Calculate the length of an arc

An arc is a part of the circumference of a circle bounded by two radii. For any given circle, the length of the arc is directly proportional to the angle subtended at the centre. As the angle increases, the arc increases as well.

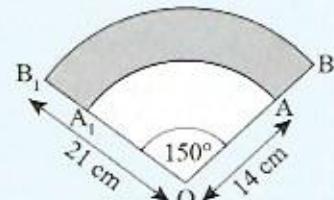
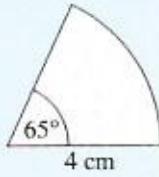
In the diagram alongside, O is the centre of the circle. The angle between the two radii is θ and the length of the arc, PQ , formed by the radii, is l .

We can express the length of the arc as a fraction of the circumference of the whole circle as $\frac{\theta}{360^\circ}$, so the length of the arc is given by the formula: $l = \frac{\theta}{360^\circ} \times 2\pi r$.



Worked example 1

- In the diagram alongside, a sector has a radius of 4 cm and it subtends an angle of 65° at the centre of the circle.
 - Calculate the area of the sector.
 - Calculate the length of the arc of the sector.
- A car's windscreen wiper left a part of a windscreen unwiped, as shown in the diagram alongside.
 - Calculate the length of arc BB_1 .
 - Calculate the length of arc AA_1 .
 - Calculate the area of the shaded part of the windscreen.



Worked example 1 (continued)

Answers

1 a) $A = \frac{\theta}{360^\circ} \times \pi r^2$
 $= \frac{65^\circ}{360^\circ} \times \pi \times 4^2$
 $= 9.08 \text{ cm}^2$

b) $l = \frac{\theta}{360^\circ} \times 2\pi r$
 $= \frac{65^\circ}{360^\circ} \times 2 \times \pi \times 4$
 $= 4.54 \text{ cm}$

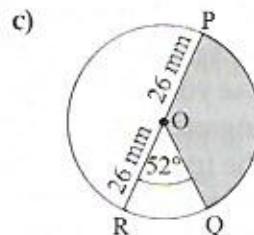
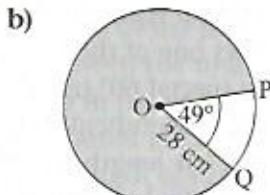
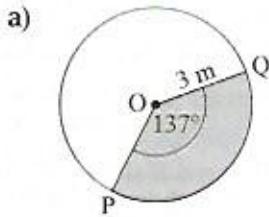
2 a) $BB_1 = \frac{\theta}{360^\circ} \times 2\pi r$
 $= \frac{150^\circ}{360^\circ} \times 2 \times \pi \times 21$
 $= 54.98 \text{ cm}$

b) $AA_1 = \frac{\theta}{360^\circ} \times 2\pi r$
 $= \frac{150^\circ}{360^\circ} \times 2 \times \pi \times 14$
 $= 36.65 \text{ cm}$

c) $A = \text{area of sector } OBB_1 - \text{area of sector } OAA_1$
 $= \frac{\theta}{360^\circ} \times \pi R^2 - \frac{\theta}{360^\circ} \times \pi r^2$
 $= \frac{150^\circ}{360^\circ} \times \pi \times 21^2 - \frac{150^\circ}{360^\circ} \times \pi \times 14^2$
 $= 320.70 \text{ cm}^2$

Activity 1

- Calculate the length of the arc of each of the following sectors.
 - The angle of the sector is 35° and the radius of the circle is 20 cm.
 - The angle of the sector is 250° and the radius of the circle is 10 m.
 - The angle of the sector is 168° and the radius of the circle is 34 mm.
 - The angle of the sector is 9° and the radius of the circle is 3 km.
- Calculate the area of each of the sectors in Question 1.
- Calculate the area of the shaded regions in the following diagrams:



Calculate the surface area of three-dimensional figures

The **total surface area (TSA)** of an object is the sum of the areas of all the faces of the object.

From earlier grades, you already know how to calculate the surface area of cubes, cuboids, cylinders and triangular prisms. In this sub-topic, you will calculate the surface area of pyramids and cones.

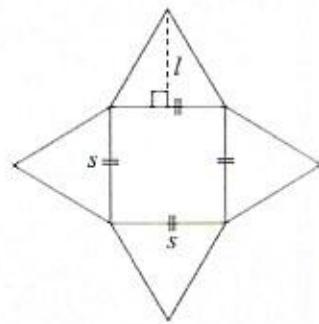
Calculate the surface area of a pyramid

Alongside is a net of a square-based pyramid.

Each side of the square base is s and the slant height of the pyramid is l .

The total surface area of the pyramid is found by adding the areas of the four triangles to the area of the square base.

$$\begin{aligned}\text{TSA} &= 4\left(\frac{1}{2}sl\right) + s^2 \\ &= 2sl + s^2 \\ &= s(2l + s)\end{aligned}$$

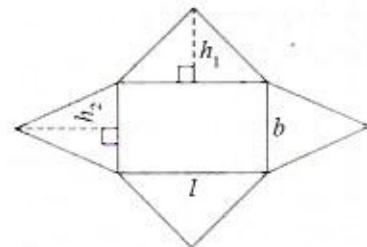


The formula for calculating the total surface area of a pyramid varies according to the shape of the base of the pyramid.

Activity 2

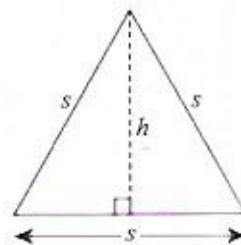
Work in small groups for this activity.

- 1 The diagram alongside shows the net of a rectangular-based pyramid. The rectangular base has a length and breadth of l and b respectively. The triangular faces have perpendicular heights of h_1 and h_2 respectively.



Use this net to derive a formula for the total surface area of a rectangular-based pyramid.

- 2 A regular tetrahedron is a triangular-based pyramid of which all four faces are identical equilateral triangles. The diagram on the right shows one of these faces.
 - a) Use your knowledge of the special 60° triangle in trigonometry to express the perpendicular height of the triangle, h , in terms of s , the length of its sides.
 - b) Now derive a formula for the total surface area of regular tetrahedron.



New words

total surface area: the sum of the areas of all the faces of a three-dimensional object

TSA: the abbreviation of "total surface area"

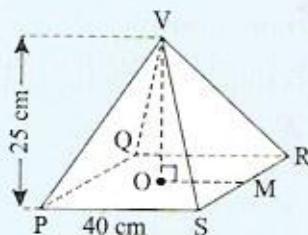
pyramid: a three-dimensional object that has a polygon as a base and all its other faces are triangles that meet at a common vertex

regular tetrahedron: a triangular-based pyramid of which all four faces are identical equilateral triangles

Worked example 2

The diagram alongside shows a pyramid PQRS on a square base with sides of 40 cm. The vertex V is 25 cm vertically above O, where O is the point of intersection of the diagonals PR and QS of the square.

- Calculate the length of VM, the slant height of the pyramid.
- Calculate the total surface area of the pyramid.

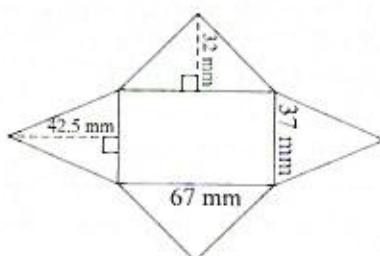


Answers

- $OM = \frac{1}{2}PS = \frac{1}{2} \times 40 = 20 \text{ cm}$
 $\therefore VM = \sqrt{OM^2 + OV^2}$ (Theorem of Pythagoras)
 $= \sqrt{20^2 + 25^2}$
 $= 32.02 \text{ cm}$
- $TSA = s(2l + s)$
 $= 40(2 \times 32.02 + 40)$
 $= 4161.6 \text{ cm}^2$

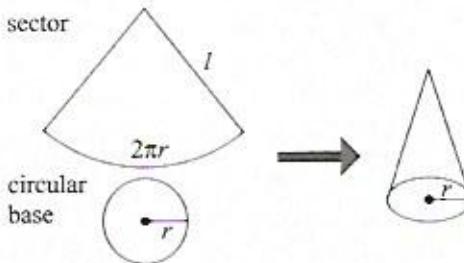
Activity 3

- Calculate the total surface area of a square-based pyramid whose slant height is 149 m and each side of the base is 230 m.
- The area of the base of a square-based pyramid is 100 mm^2 and its slant height is 25 mm.
 - Calculate the total surface area of the pyramid.
 - Express the relationship between the total surface area of the pyramid and the area of its base as a ratio in its simplest form.
- If the total surface area of a pyramid is 21 cm^2 and the slant height of the pyramid is 2 cm, calculate the length of one side of the square base of the pyramid.
- The net of a rectangular-based pyramid is shown alongside.
 - Use the formula that you derived in Question 1 of Activity 2 to calculate the total surface area of the pyramid.
 - Calculate the perpendicular height of the pyramid, correct to the nearest millimetre.
- A regular tetrahedron has sides of 142 mm. Use the formula that you derived in Question 2 of Activity 2 to calculate its total surface area.



Calculate the surface area of a cone

A cone consists of a sector whose radius is equal to l , the slant edge of the cone. The arc of the sector is equal to the circumference of the circular base of the cone.



New word

cone: a three-dimensional object that has a circular base and one vertex

To calculate the total surface area of a cone, we need to add the area of the curved surface and the area of the circle.

We know that the length of the arc of the sector = $2\pi r$, because the arc of the sector is equal to the circumference of the base circle.

The fraction that the sector is of the whole circle with radius l = $\frac{2\pi r}{2\pi l} = \frac{r}{l}$.

So, the area of the curved surface = $\frac{r}{l} \times \pi l^2 = \pi r l$.

The area of the base circle = πr^2 .

So, the total surface area of the cone is given by the formula: $TSA = \pi r^2 + \pi r l = \pi r(r + l)$.

Worked example 3

A cone has a slant height of 12 cm and the radius of its circular base is 5 cm. Calculate the total surface area of the cone.

Answer

$$\begin{aligned} l &= \sqrt{r^2 + h^2} \\ &= \sqrt{5^2 + 12^2} = 13 \text{ cm} \end{aligned}$$

$$\begin{aligned} TSA &= \pi r(r + l) \\ &= \pi \times 5(5 + 13) = 282.74 \text{ cm}^2 \end{aligned}$$

Activity 4

- Calculate the total surface area of each of the following cones:
 - the slant height is 1.1 m and the radius of its base is 0.75 m
 - the slant height is 45 cm and the radius of its base is 28 cm.
 - the slant height is 130 mm and the diameter of its base is 245 mm.
 - the slant height is 6.3 m and the diameter of its base is 7.2 m.
- The slant height and the radius of the base of a cone are in the ratio 2 : 1. If the radius of the cone is 145 mm, calculate the total surface area of the cone.
- The slant height of a cone is equal to the diameter of the base of the cone. If the slant height of the cone is 32 cm, calculate the total surface area of the cone.

Calculate the volume of three-dimensional figures

The volume of a three-dimensional object is the amount of space occupied by the object. From earlier grades, you already know how to calculate the volume of cubes, cuboids, cylinders and triangular prisms. In this sub-topic, you will calculate the volume of cones, rectangular and triangular pyramids and frustums of cones and pyramids.

New word

volume: the amount of space that a three-dimensional object occupies

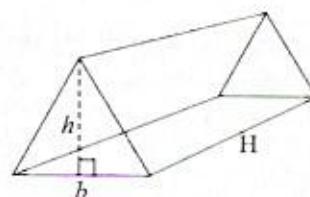
Calculate the volume of a prism

The volume of any **prism** is given by the formula:

$$V = \text{area of the base} \times \text{height of prism}$$

So, for example, the volume of the triangular prism alongside is:

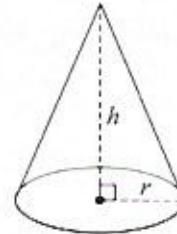
$$\begin{aligned} V &= \text{area of } \triangle \times \text{height of prism} \\ &= \frac{1}{2}bh \times H = \frac{1}{2}bhH \end{aligned}$$



Calculate the volume of a cone

The volume of a cone is given by the formula: $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the circular base of the cone and h is the perpendicular height of the cone.

This can also be written as follows: $V = \frac{1}{3} \times \text{area of base} \times h$, where h is the perpendicular height of the cone. This second formula is useful if the area of the base of the cone is given.



Worked example 4

Calculate the volume of a cone of which the circular base has a diameter of 30 cm and the perpendicular height is 20 cm.

Answer

$$\begin{aligned} V &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3} \times \pi \times 15^2 \times 20 \\ &= 4712.39 \text{ cm}^3 \end{aligned}$$

Activity 5

- Calculate the volume of each of the following cones, if:
 - the area of the base is 25 cm^2 and the perpendicular height is 4 cm.
 - the area of the base is 1 m^2 and the perpendicular height is 75 cm.
 - the area of the base is 316 cm^2 and the perpendicular height is 30 cm.
 - the area of the base is 52 mm^2 and the perpendicular height is 15 mm.

Activity 5 (continued)

2 Calculate the volume of each of the following cones, if:

- the radius of the circular base is 3 m and the perpendicular height is 1.5 m.
- the radius of the circular base is 41 cm and the perpendicular height is 14 cm.
- the diameter of the circular base is 68 mm and the perpendicular height is 132 mm.
- the diameter of the circular base is 50 cm and the perpendicular height is 20 cm.

Calculate the volume of a pyramid

The volume of a pyramid is given by the formula: $V = \frac{1}{3} \times \text{area of base} \times h$, where h is the perpendicular height of the pyramid.

Worked example 5

Calculate the volume of a rectangular pyramid of which the perpendicular height is 7.5 cm and the base has a length of 30 cm and a breadth of 12.5 cm.

Answer

$$\begin{aligned}V &= \frac{1}{3} \times \text{area of base} \times h \\&= \frac{1}{3} \times 30 \times 7.5 \times 12.5 = 937.5 \text{ cm}^3\end{aligned}$$

Activity 6

- Calculate the volume of each of the following pyramids, if:
 - the area of the base is 40 cm^2 and the perpendicular height is 8 cm.
 - the area of the base is 90 m^2 and the perpendicular height is 10.5 m.
 - the area of the base is 49 mm^2 and the perpendicular height is 9 mm.
 - the area of the base is 820 mm^2 and the perpendicular height is 3 cm.
- Calculate the volume of each of the following pyramids, if:
 - the dimensions of the rectangular base are 15 cm by 10 cm and the perpendicular height is 5 cm.
 - the square base has sides of 26 mm and the perpendicular height is 12 mm.
 - the base is a triangle with sides of 3 cm, 4 cm and 5 cm and the perpendicular height of the pyramid is 6 cm.

Calculate the volume of a frustum

A **frustum** of a cone or a pyramid is that part of the cone or pyramid that remains if the top part has been cut off. The cut is always made parallel to the base of the cone or pyramid.

A frustum with a triangular, square or rectangular cross-section is the frustum of a pyramid. A frustum with a circular cross-section is the frustum of a cone.

New word

frustum: that part of a cone or pyramid that remains if the top part has been cut off

Worked example 6

The diagram alongside shows a frustum with a circular base and a circular cross-section. The radius of the base is 40 cm, the radius of the cross-section is 20 cm and the height of the frustum is 60 cm. Calculate the volume of the frustum.

Answer

This is the frustum of a cone, because its cross-section is circular.

The cut-off part of the cone as well as the complete cone are shown alongside, where x is the height of the cut-off part of the cone.

We can use our knowledge of proportion to solve for x .

$$\frac{x}{20} = \frac{60+x}{40}$$

$$\therefore 40x = 20(60+x)$$

$$\therefore 40x = 1200 + 20x$$

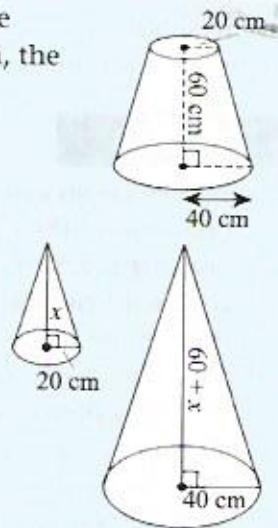
$$\therefore 20x = 1200$$

$$\therefore x = 60 \text{ cm}$$

So, the height of the larger cone is $60 + 60 = 120$ cm and that of the smaller cone is 60 cm.

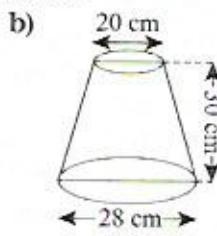
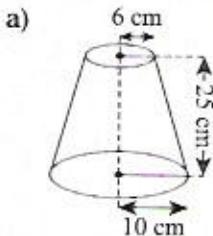
We can now calculate the volume of the frustum by subtracting the volume of the smaller cone from the volume of the larger cone.

$$\begin{aligned} \text{Volume of frustum} &= \frac{1}{3}\pi R^2 H - \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi(R^2 H - r^2 h) \\ &= \frac{1}{3} \times \pi[(40^2 \times 120) - (20^2 \times 60)] \\ &= 175\,952.19 \text{ cm}^3 \end{aligned}$$



Activity 7

1 Calculate the volume of the frustums below.



2 The frustum of a square-based pyramid has the following dimensions: each side of the square base is 5 cm long, each side of the square cross-section is 3 cm long and the perpendicular height of the frustum is 4 cm. Calculate the volume of the frustum.

Solve problems involving area and volume

In this sub-topic, you will solve problems involving area and volume in a number of different real-life contexts.

Activity 8

- 1 The ice cream cone alongside has these dimensions: the radius of the circular opening is 2.5 cm and the slant height of the cone is 13 cm. Calculate the surface area of the outside of the cone.
- 2 The traffic cone shown alongside has the following dimensions: the radius of the circular base is 10 cm and the height of the cone is 40 cm. Calculate the volume of the air inside the cone.
- 3 The photograph alongside shows the Great Pyramid at Giza in Egypt. It is the oldest of the Seven Wonders of the Ancient World. It is also the only one to remain more or less intact. The perpendicular height of this pyramid is 138.8 m and the side length of its square base is 230.4 m. Calculate the volume of this pyramid.
- 4 A child's stacking toy is in the shape of a square-based pyramid, as shown alongside. It consists of seven individual pieces, each of which is 4 cm thick. One side of the base of the bottom piece measures 21.2 cm.
 - a) Calculate the total surface area of the assembled toy.
 - b) If a child has assembled the bottom four layers of the toy, calculate the volume of the part of the toy that he has assembled.
- 5 A glass paperweight is in the shape of a rectangular-based pyramid. Inside the paperweight is a small copper cone. The pyramid has a height of 4 cm and its base is 6 cm by 3.5 cm. The base of the cone has a radius of 1.25 cm and the perpendicular height of the cone is 2 cm.
 - a) Calculate the volume of the cone.
 - b) Calculate the volume of the paperweight.
 - c) Calculate the volume of glass in the paperweight.
 - d) Express the volume of the cone as a percentage of the volume of the paperweight.



Summary

Area

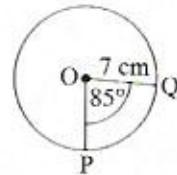
- The area of a sector of a circle is: $A = \frac{\theta}{360^\circ} \times \pi r^2$, where θ is the angle subtended by the sector at the centre of the circle and r is the radius of the circle.
- The length of an arc of a sector of a circle is: $l = \frac{\theta}{360^\circ} \times 2\pi r$, where θ is the angle subtended by the sector at the centre of the circle and r is the radius of the circle.
- The total surface area (TSA) of an object is the sum of the areas of all the faces of the object.
- A pyramid is a three-dimensional object that has a polygon as a base and all its other faces are triangles that meet at a common vertex.
- The total surface area of a square-based pyramid is: $TSA = s(2l + s)$, where s is the length of one side of the base and l is the slant height of the pyramid.
- A regular tetrahedron is a triangular-based pyramid of which all four faces are identical equilateral triangles.
- A cone is a three-dimensional object that has a circular base and one vertex.
- The total surface area of a cone is: $TSA = \pi r^2 + \pi r l = \pi r(r + l)$, where r is the radius of the base of the cone and l is the slant height of the cone.

Volume

- The volume of a prism is: $V = \text{area of base} \times \text{height of prism}$.
- The volume of a cone is: $V = \frac{1}{3} \pi r^2 h$, where r is the radius of the circular base of the cone and h is the perpendicular height of the cone. Alternatively: $V = \frac{1}{3} \times \text{area of base} \times h$, where h is the perpendicular height of the cone.
- The volume of a pyramid is $V = \frac{1}{3} \times \text{area of base} \times h$, where h is the perpendicular height of the pyramid.
- A frustum of a cone or a pyramid is that part of the cone or pyramid that remains if the top part has been cut off.
- A frustum with a triangular, square or rectangular cross-section is the frustum of a pyramid. A frustum with a circular cross-section is the frustum of a cone.

Revision exercises (remedial)

- In the diagram alongside, arc PQ subtends an angle of 85° at the centre of the circle.
 - Calculate the area of the minor sector POQ. (2)
 - Calculate the length of the minor arc PQ. (2)
- Use the formula $V = \frac{1}{3} \times \text{area of base} \times h$ to calculate the volume of a square-based pyramid, if the sides of the square base are 10 cm and the perpendicular height of the pyramid is 15 cm. (2)



Revision exercises

3 The angle of a sector is 130° and the radius of the circle is 9.2 cm. (2)

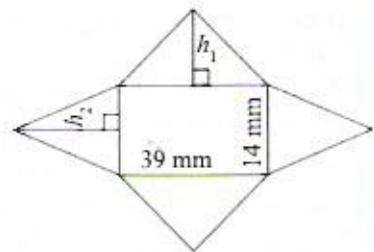
- Calculate the area of the sector. (2)
- Calculate the length of the arc of the sector. (2)

4 The circular base of a cone has a radius of 17 cm and a perpendicular height of 34 cm. (2)

- Calculate the slant height of the cone. (2)
- Calculate the total surface area of the cone. (2)
- Calculate the volume of the cone. (2)

5 The net of a rectangular-based pyramid is shown alongside. Its base has sides of 39 mm and 14 mm and its perpendicular height is 21 mm. (4)

- Calculate the lengths of h_1 and h_2 . (3)
- Calculate the total surface area of the pyramid. (3)
- Calculate the volume of the pyramid. (2)



Total marks: 25

Assessment exercises

1 Calculate the angle subtended at the centre of a circle if the area of the sector is 20 cm^2 and the radius of the circle is 8 cm. (4)

2 A regular tetrahedron has sides of 25 mm. Calculate the total surface area of the tetrahedron. (3)

3 The base of the frustum of a cone has a radius of 8.4 cm. The cross-section of the frustum has a radius of 6 cm. The perpendicular height of the frustum is 9 cm. (6)

- Calculate the volume of the frustum. (6)
- Calculate the total surface area of the frustum. (6)

4 A jeweller has a cone-shaped piece of metal. The base of the cone has a radius of 1 cm and a perpendicular height of 2 cm. He melts the metal down to make a square-based pyramid. The base has sides of 1 cm. Calculate the perpendicular height of the pyramid. (6)

5 A cone has a slant length, l , and the radius of its base is r . (2)

- Express the perpendicular height, h , of the cone in terms of l and r . (2)
- Hence, express the ratio of the total surface area of the cone to its volume as a unit ratio in the form 1 : _____. (3)

Total marks: 30

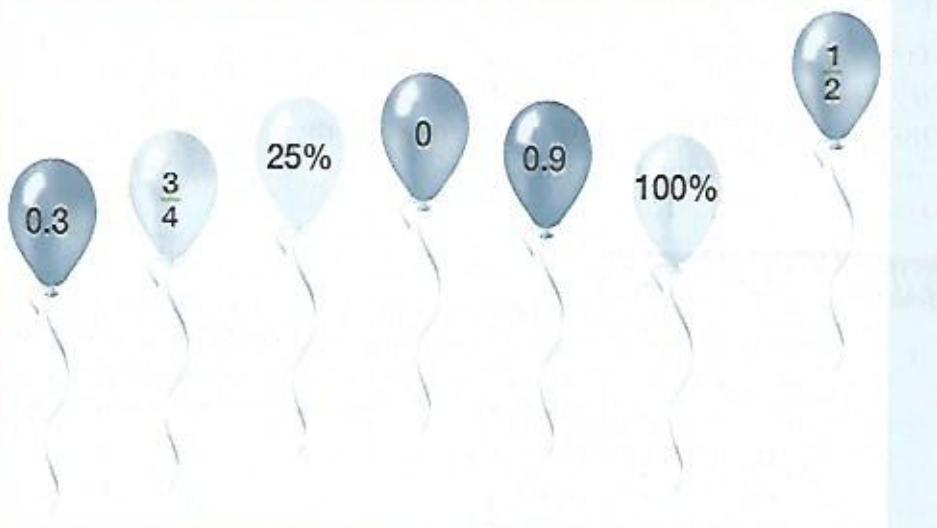


Sub-topics	Specific Outcomes
Laws of probability	<ul style="list-style-type: none">Compute probabilities using the laws of probability
Tree diagrams and grids	<ul style="list-style-type: none">Calculate probabilities using tree diagrams and gridsCalculate probabilities of mutually exclusive events and compound eventsFind probabilities of independent eventsApply probability to real-life problems

Starter activity

Work in pairs for this activity.

- 1 Arrange the numbers on the balloons below in order from the least likely to the most likely.



- 2 What number above is the probability of an event that:
 - has no chance of happening?
 - is certain to happen?
 - has a fifty-fifty chance of happening?

You have already worked with probability in Grade 9. In this sub-topic, we will first revise what you already know and then we will apply the laws of probability to single events.

Compute probabilities using the laws of probability

Below is a brief revision of the some of the terms that we use in probability, as well as some of the laws of probability.

- The **probability** of something happening is the likelihood of that thing happening.
- A probability is always expressed as a number from 0 to 1.
- If we conduct an experiment once, this is called a **trial**.
- A result of an experiment is an **outcome** of the experiment.
- An **event** is one particular outcome in which we are interested.
- The total number of possible outcomes of a trial form a **sample space**.
- If A is one possible outcome of a trial, then the probability of A, denoted by $P(A)$, is given by:

$$P(A) = \frac{\text{Number of outcomes favourable to } A}{\text{Total number of possible outcomes in the trial}}$$

- If A and B are the only possible outcomes of a trial then $P(A) + P(B) = 1$. Similarly if A, B and C are the only possible outcomes of a trial, then

$$P(A) + P(B) + P(C) = 1.$$

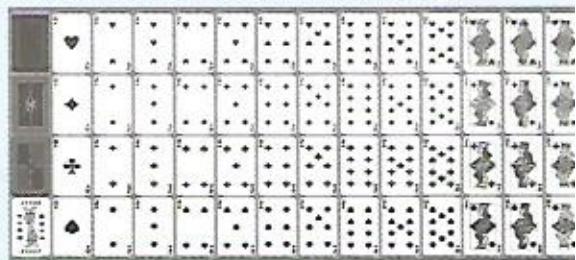
New words

probability: likelihood
trial: one performance of an experiment
outcome: a result of an experiment
event: one particular outcome in which we are interested
sample space: the total number of possible outcomes of a trial

Worked example 1

1 In a cup final, the probability that Manchester United will win the cup is $\frac{3}{5}$. Find the probability that Manchester United will not win the cup.

2 In a standard pack of playing cards, there are four suits: hearts, spades, diamonds and clubs, as shown alongside. Each suit has a 2, 3, 4, 5, 6, 7, 8, 9, 10, a Jack, a Queen, a King and an Ace. The Ace and the numbers 2 to 10 are called number cards and the Jack, Queen and King are called picture cards. So, each suit has 13 cards and there are 52 cards altogether.



Worked example 1 (continued)

A card is drawn from a well-shuffled pack of cards. What is the probability that it is:

- a) a red card
- b) a 7
- c) the King of diamonds
- d) not the King of diamonds?

Answers

1 In a cup final, there are only two possible outcomes: win (W) or lose (L).

$$P(W) + P(L) = 1$$

$$\therefore \frac{3}{5} + P(L) = 1$$

$$\therefore P(L) = 1 - \frac{3}{5} = \frac{2}{5}$$

2 a) Out of 52 cards, 26 cards are red (hearts and diamonds).

$$P(\text{red}) = \frac{26}{52} = \frac{1}{2}$$

b) Out of 52 cards, four cards are a 7.

$$P(7) = \frac{\text{Number of outcomes favourable to 7}}{\text{Total number of possible outcomes}} = \frac{4}{52} = \frac{1}{13}$$

c) Out of 52 cards, only one card is the King of diamonds.

$$\therefore P(\text{King of diamonds}) = \frac{1}{52}$$

d) Out of 52 cards, 51 cards are not the King of diamonds.

$$\therefore P(\text{not the King of diamonds}) = \frac{51}{52}$$

Alternatively; $P(\text{not the King of diamonds}) + P(\text{King of diamonds}) = 1$

$$\therefore P(\text{not the King of diamonds}) = 1 - P(\text{King of diamonds}) = 1 - \frac{1}{52} = \frac{51}{52}$$

$$\therefore P(\text{not the King of diamonds}) = \frac{51}{52}$$

Activity 1

1 The probability that a soccer team will win a match is $\frac{2}{3}$. What is the probability that the team will not win the match?

2 If the probability of an event occurring is $\frac{p}{q}$, what is the probability of the event not occurring?

3 What is the probability that, when a fair coin is tossed, it shows tails?

4 What is the probability that, when a die is rolled, the score is 2?

5 A bag contains 15 identical marbles of which 6 are blue, 4 are yellow and the rest are red. If a marble is chosen at random from the bag, find the probability that:

- a) it is yellow
- b) it is blue
- c) it is red
- d) it is yellow or blue.

6 If a die is rolled, find the probability that:

- a) the score is less than 3
- b) the score is a prime number
- c) the score is a multiple of 2
- d) the score is 2 or 3.

Activity 1 (continued)

7 If a number is chosen at random from the following set: {1, 2, 3, 4, 5, ... 30}, find the probability that it is:

- 8
- a factor of 12
- a multiple of 10
- not a multiple of 10.

8 A letter is chosen at random from the word MATHEMATICS. Find the probability that it is:

- an A
- a vowel
- a consonant
- an M or a T
- not an M or a T.

9 A box contains x green beads and y red beads. Given that the probability of choosing a green bead from the box is $\frac{3}{4}$:

- express x in terms of y
- find x if $y = 2$
- find the probability of choosing a red bead if $y = 2$.

If in a trial the probability of an event A is $P(A)$ and the trial is carried out n times, then the expected number of events A is given by $n \times P(A)$.

Worked example 2

In a certain community, the probability that a person chosen at random is left-handed is $\frac{1}{10}$. Find the expected number of left-handed people in a sample of 2 000 people randomly chosen from the community.

Answer

The expected number = $n \times P(A) = 2\ 000 \times \frac{1}{10} = 200$ left-handed people.

Activity 2

1 In a school, the probability that a learner chosen at random wears spectacles is $\frac{2}{25}$. Find the expected number of learners who wear spectacles in a random sample of 350 learners in the school.

2 When recruiting a goalkeeper for a season, the manager of a team considers the probability of the goalkeeper saving a penalty. Three goalkeepers A, B and C are on trial for this position. The probabilities of saving a penalty are as follows:
A: $\frac{7}{15}$, B: $\frac{5}{11}$ and C: $\frac{13}{30}$.

- Which goalkeeper has the highest chance of being selected?
- Which goalkeeper has the smallest chance of being selected?
- To confirm his choice, the manager tests each of them with 330 penalties over a period of time. How many of these penalties is each goalkeeper expected to save?

Tree diagrams and grids

In the previous sub-topic, you worked with simple events. In this sub-topic, you will work with compound events. **Compound events** are two or more events that together form one trial of an experiment.

Calculate probabilities using tree diagrams

Tree diagrams are used to solve problems involving compound events. The following examples illustrate the use of tree diagrams to solve such problems.

New words

compound events: two or more events that together form one trial of an experiment

tree diagram: a diagram that is used to solve problems involving compound events

Worked example 3

- 1 A coin is tossed twice.
 - Draw a tree diagram to show all the possible outcomes and corresponding probabilities.
 - Use your tree diagram to find the probability of getting:
 - two heads
 - at least one heads
 - a tails and a heads.
- 2 A box contains 5 red counters and 7 blue counters. Two counters are drawn in succession at random without replacement.
 - Draw a tree diagram to show all the possible outcomes and corresponding probabilities.
 - Use your tree diagram to find the probability of getting:
 - two red counters
 - at least one red counter.

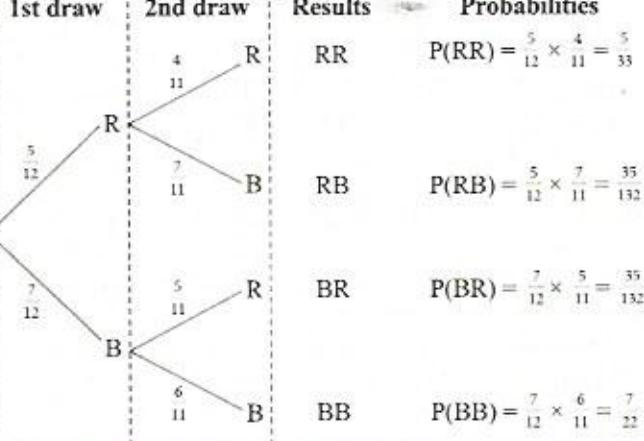
Answers

1 a)	1st toss	2nd toss	Results	Probabilities
		H	HH	$P(HH) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
		T	HT	$P(HT) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
	H		TH	$P(TH) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
	T		TT	$P(TT) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

b) (i) $P(\text{two heads}) = P(HH) = \frac{1}{4}$
 (ii) $P(\text{at least one heads}) = P(HH) + P(HT) + P(TH) = \frac{3}{4}$
 (iii) $P(\text{a tails and a heads}) = P(HT) + P(TH) = \frac{1}{2}$

Worked example 3 (continued)

2 a)



b) (i) $P(\text{two red counters}) = P(RR) = \frac{5}{33}$

$$(ii) P(\text{at least one red counter}) = 1 - P(\text{BB}) = 1 - \frac{7}{22} = \frac{15}{22}$$

Activity 3

1 A coin is tossed three times.

- Draw a tree diagram to show all the possible outcomes and probabilities.
- Use your tree diagram to find the probability of getting:
 - three heads
 - at least one heads
 - two tails and a head

2 A box contains 5 red and 7 pink beads. Two beads are drawn and not replaced.

- Draw a tree diagram to show all the possible outcomes and probabilities.
- Use your tree diagram to find the probability of drawing:
 - two red beads
 - only one red bead
 - two pink beads
 - at least one pink bead.

Calculate probabilities using grids

A probability grid is a table that shows all the possible outcomes of a trial that consists of two events.

Worked example 4

A red die and a blue die are rolled.

a) Construct a probability grid to show all the possible outcomes.
b) Use your probability grid to find the probability of getting:
(i) the same score on both dice
(ii) a sum of 9 if the results are added together
(iii) a product of 36 if the results are multiplied together.

Worked example 4 (continued)

Answers

a) The probability grid alongside shows all of the 36 possible outcomes.

b) (i) The same score on both dice are shown on the diagonal from (1, 1) to (6, 6).

$$\therefore P(\text{same score on both dice}) = \frac{6}{36} = \frac{1}{6}$$

(ii) A sum of 9 is obtained in the following results:

(3, 6), (4, 5), (5, 4) and (6, 3).

$$\therefore P(\text{sum of 9}) = \frac{4}{36} = \frac{1}{9}$$

(iii) The only outcome that will result in a product of 36 is (6, 6).

$$\therefore P(\text{product of 36}) = \frac{1}{36}$$

		Red die					
		1	2	3	4	5	6
Blue die	1	1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
	2	2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
	3	3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
	4	4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
	5	5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
	6	6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

Activity 4

1 A coin is tossed and a die is rolled.

a) Construct a probability grid to show all the possible outcomes.

b) Use your probability grid to find the probability of getting:

(i) heads and any number (ii) tails and an odd number
 (iii) heads and a 6.

2 Two dice, A and B, are rolled and the scores are added together.

a) Construct a probability grid to show all the possible outcomes.

b) Use your probability grid to find the probability that the sum will be:

(i) greater than 7 (ii) a square number (iii) a multiple of 4.

Calculate probabilities of mutually exclusive events and compound events

Thus far, you have used tree diagrams and probability grids to calculate the probabilities of compound events. However, this can be time-consuming and cumbersome. We are now going to use formulae to calculate the probability of compound events. Events A and B are **mutually exclusive** if they cannot happen at the same time. For example, if you roll a die, you cannot get a 6 and a 1 at the same time. For any two events A and B that are mutually exclusive:

$$P(A \text{ or } B) = P(A) + P(B).$$

This is called the **addition rule for mutually exclusive events**.

New words

mutually exclusive events: events that cannot happen at the same time

addition rule for mutually exclusive events: $P(A \text{ or } B) = P(A) + P(B)$

Worked example 5

In a class test, the probability that Mubita will be first in the class is $\frac{3}{14}$ and the probability that Nosiku will be first in the class is $\frac{1}{5}$. Find the probability that Mubita or Nosiku will be first in the class, assuming that there will not be a tie.

Answer

These events are mutually exclusive. Let M and N represent the events that Mubita and Nosiku are first in the class respectively.

$$\therefore P(M \text{ or } N) = P(M) + P(N) = \frac{3}{14} + \frac{1}{5} = \frac{29}{70}$$

Activity 5

- 1 In a sample of 100 light bulbs, the probability that none of them is defective is 0.2 and the probability that exactly one light bulb is defective is 0.35.
 - a) If A is the event (no light bulbs are defective) and B is the event (one light bulb is defective), explain why A and B are mutually exclusive events.
 - b) Calculate the probability that at most one light bulb is defective.
 - c) Calculate the probability that more than one light bulb is defective.
- 2 At a news stand one Saturday morning, 11 people bought newspaper A, 14 bought newspaper B and 5 bought newspaper C. Nobody bought more than one newspaper. If one customer was chosen from the group at random, calculate the probability that they bought:
 - a) newspaper A
 - b) newspaper C
 - c) newspaper A or newspaper B
 - d) newspaper B or newspaper C

Find probabilities of independent events

Events are **independent** if the occurrence of one does not influence the occurrence of the other in any way. For example if a coin is tossed and a die is rolled, the result on the coin does not influence the result on the die. For any two events A and B that are independent:

$$P(A \text{ and } B) = P(A) \times P(B)$$

This is called the **multiplication rule for independent events**.

New words

independent events:
events that cannot influence one another

multiplication rule for independent events:
 $P(A \text{ and } B) = P(A) \times P(B)$

Worked example 6

If a coin is tossed and a die is rolled, find the probability of getting tails and a 6.

Answer

$$P(\text{Tails}) = \frac{1}{2} \text{ and } P(6) = \frac{1}{6} \therefore P(\text{Tails and } 6) = P(\text{T}) \times P(6) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$$

Activity 6

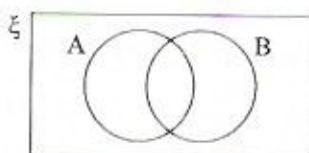
- Chongo is choosing what to wear. He has three shirts (red, white and blue) and two pairs of pants (black and blue). If he chooses a shirt and a pair of pants at random, what is the probability that he chooses a white shirt and a black pair of pants?
- In a Grade 11 class there is a pair of twins, Kombe and Milika. There are 24 boys in the class and 26 girls. If the teacher chooses a boy and a girl from the class at random, what is the probability that the twins will be chosen?

We will explore these ideas further using set theory. For any two intersecting sets A and B, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$. If the equation is divided throughout by $n(E)$, the total number of members in the universal set (ξ), we get the associated probability:

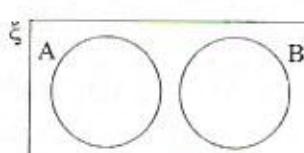
$$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

If events A and B are mutually exclusive, $A \cap B = 0$.

$$\therefore P(A \cup B) = P(A) + P(B), \text{ which is the addition rule for mutually exclusive events.}$$



A and B are not mutually exclusive, so $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.



A and B are mutually exclusive, so $P(A \cup B) = P(A) + P(B)$.

Worked example 7

If a number is chosen randomly from the set of numbers {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, find the probability that the number is:

- an even number, or an odd number greater than 6
- an even number or a prime number.

Answers

- Let $A = \{2, 4, 6, 8, 10\}$ and $B = \{7, 9\}$. $A \cap B = \{\}$, so A and B are mutually exclusive.
 $\therefore P(A \cup B) = P(A) + P(B) = \frac{5}{10} + \frac{2}{10} = \frac{7}{10}$
- Let $A = \{2, 4, 6, 8, 10\}$ and $B = \{2, 3, 5, 7\}$. $A \cap B = \{2\}$, so A and B are not mutually exclusive.
 $\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{5}{10} + \frac{4}{10} - \frac{1}{10} = \frac{8}{10} = \frac{4}{5}$

You already know that discrete data is data that can be counted and continuous data is data that cannot be counted. In the same way, we have a **discrete sample space** and a **continuous sample space**.

An example of a discrete sample space is the set $S = \{\text{all integers between 30 and 40}\}$. An example of a continuous sample space is the set $S = \{\text{all real numbers between 30 and 40}\}$.

New words

discrete sample space: a sample space containing a finite number of outcomes
continuous sample space: a sample space containing an infinite number of outcomes

Activity 7

- Given that $S = \{1, 3, 4, 7, 9, 11, 13, 16, 19\}$, $A = \{1, 4, 11\}$ and $B = \{3, 9, 11, 13\}$. If a number, x , is chosen at random from S , find the probability that:
 - $x \in A$
 - $x \in B$
 - $x \in (A \cap B)$
 - $x \in (A \cup B)$
- A point is plotted at random on the Cartesian plane. Assume that the point does not fall on either of the axes. If $A = \{\text{all points in the 1st quadrant}\}$, $B = \{\text{all points in the 2nd quadrant}\}$, $C = \{\text{all points in the 3rd quadrant}\}$ and $D = \{\text{all points in the 4th quadrant}\}$, calculate:
 - $P(A)$
 - $P(A \cap B)$
 - $P(B \cup C \cup D)$
 - $P(C \text{ or } D)$
- Explain why the set of all points on the Cartesian plane is a continuous sample space.

Apply probability to real-life problems

You will now solve real-life problems in a variety of contexts.

Activity 8

- 60% of the people in a certain community eat caterpillars.
 - Three people are chosen at random from the community. Calculate the probability that all three people eat caterpillars.
 - In a sample of 3 000 people chosen randomly from the community, how many are expected to eat caterpillars?
- The probability that Tina goes to school on any school day is 0.9.
 - Draw a tree diagram to show her attendance or non-attendance over two consecutive school days.
 - Find the probability that she attends school on one day only.
- In a certain village, there are 580 teenagers. Of these, 280 are boys. 123 girls and 154 boys are sexually active. A girl and a boy are chosen at random. Calculate the following probabilities and express them as percentages.
 - The boy is not sexually active.
 - The girl is not sexually active.
 - Both are sexually active.
- The probability that a maize seed germinates is $\frac{9}{10}$ and if it germinates, the probability that it survives a drought is $\frac{1}{5}$.
 - Find the probability that a seed germinates and survives a drought.
 - If 400 maize seeds are planted, how many are expected to germinate and survive a drought?

Summary

Laws of probability

- The probability of something happening is the likelihood of that thing happening.
- A probability is always expressed as a number from 0 to 1.
- If we conduct an experiment once, this is called a trial.
- A result of an experiment is an outcome of the experiment.
- An event is one particular outcome in which we are interested.
- The total number of possible outcomes of a trial form a sample space.
- If A is one possible outcome of a trial, then the probability of A, denoted by $P(A)$, is given by: $P(A) = \frac{\text{Number of outcomes favourable to } A}{\text{Total number of possible outcomes in the trial}}$
- If A and B are the only possible outcomes of a trial then $P(A) + P(B) = 1$.
- If in a trial the probability of an event A is $P(A)$ and the trial is carried out n times, then the expected number of events A is given by $n \times P(A)$.

Tree diagrams and grids

- Compound events are two or more events that together form one trial of an experiment.
- Tree diagrams are used to solve problems involving compound events.
- A probability grid is a table that shows all the possible outcomes of a trial that consists of two events.
- Events A and B are mutually exclusive if they cannot happen at the same time.
- For any two events A and B that are mutually exclusive: $P(A \text{ or } B) = P(A) + P(B)$.
So, $P(A \cup B) = P(A) + P(B)$.
- For any two events A and B that are not mutually exclusive,
 $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- Events are independent if the occurrence of one does not influence the occurrence of the other in any way.
- For any two events A and B that are independent: $P(A \text{ and } B) = P(A) \times P(B)$.
- A discrete sample space contains a finite number of outcomes.
- A continuous sample space contains an infinite number of outcomes.

Revision exercises (remedial)

- 1 There are 10 boys and 15 girls in a class. A learner is chosen at random from the class. Find the probability that the chosen learner is a girl. (1)
- 2 A trial has two possible outcomes, A and B. Given that $P(A) = \frac{2}{3}$, find $P(B)$. (1)

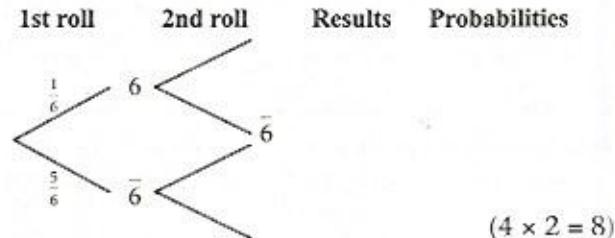
Revision exercises

3 98% of the learners in a school are right-handed.

- What is the probability that a learner chosen at random from the school is left-handed? (1)
- If 200 learners are chosen randomly from the school, how many of them can be expected to be right-handed? (2)

4 A die is rolled twice.

- Copy and complete the tree diagram below, in which 6 represents the outcome of rolling a 6, and $\bar{6}$ represents the outcome of not rolling a 6. (12)
- Use your completed tree diagram to find the probability of getting:
 - (i) two sixes
 - (ii) no sixes
 - (iii) one six
 - (iv) at least one six



(4 × 2 = 8)

Total marks: 25

Assessment exercises

1 In a certain community, 4% of the population have suffered from measles and 2% have suffered from malaria. Assume that these illnesses are unrelated. Calculate the probability that a person chosen at random from this community has suffered from:

- malaria and measles
- malaria, but not measles
- measles, but not malaria
- neither malaria nor measles. (4 × 2 = 8)

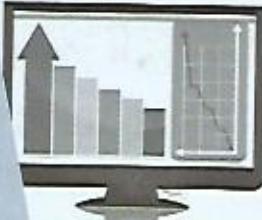
2 Two dice, A and B, are rolled and the scores are multiplied together.

- Construct a probability grid to show all the possible outcomes. (12)
- Use your probability grid to find the probability that the product will be:
 - (i) greater than 20
 - (ii) a square number
 - (iii) a multiple of 5
 - (iv) 35. (4 × 2 = 8)

3 Mbiza and Teza go bird hunting every weekend. The probability that Mbiza kills a bird on any such trip is $\frac{1}{5}$ and the probability that Teza kills a bird is $\frac{1}{3}$. On two such trips, find the probability that:

- only one bird is killed
- each one of them kills one bird
- neither makes a kill
- three birds are killed (4 × 3 = 12)

Total marks: 40



Sub-topics	Specific Outcomes
Cumulative frequency tables	<ul style="list-style-type: none"> Construct cumulative frequency tables using grouped and ungrouped data Draw cumulative frequency curves Draw relative cumulative curves
Measures of dispersion	<ul style="list-style-type: none"> Calculate the range, interquartile range and semi-interquartile range Calculate the percentiles Calculate variance and standard deviation for ungrouped and grouped data

Starter activity

Work in pairs for this activity.

1 The following are the number of people living in each house in a small village.

5, 5, 4, 10, 8, 7, 7, 6, 2, 7, 7, 9, 7, 8, 1, 10, 6, 4, 3, 3.

Find the mode, median and the mean of this data.

2 The following table shows the distribution of marks out of 10 obtained by 30 learners in a Science test.

Mark	1	2	3	4	5	6	7	8	9	10
Frequency	2	2	4	a	b	6	3	2	2	1

The mean mark for the distribution is 5.2.

- Set up and solve two simultaneous linear equations to find the values of a and b .
- Calculate the modal mark.
- Calculate the median mark.

Introduction

Statistics is one of the most useful branches of mathematics in real life. Statistics are used in many careers, including nursing, farming, weather-reporting, life and short-term insurance, education and many more. You already have some knowledge of statistics from previous grades. In this sub-topic, you will build on your knowledge as you work with cumulative frequency tables and cumulative frequency curves.

Construct cumulative frequency tables

A **cumulative frequency table** is a table that has three columns: one each for data values, frequencies and cumulative frequencies.

The **frequency** of a data value is the number of times that the data value appears in a data set.

The **cumulative frequency** of a data value is the sum of all the frequencies up to and including the frequency of that data value.

New words

cumulative frequency table: a table that has columns for data values, frequencies and cumulative frequencies

frequency (of a data value): the number of times that the data value appears in a data set

cumulative frequency (of a data value): the sum of all the frequencies up to and including the frequency of that data value

Worked example 1

The table below shows the number of days that the 52 learners in a class were absent from school in one school term.

Days absent	0	1	2	3	4	5	6	7	8	9
Frequency	2	3	6	7	6	9	11	6	0	2

Show this information in a cumulative frequency table.

Answer

In the table above, it is important to note that frequency means the number of learners who were absent for a given number of days. For example, two learners were not absent at all, three learners were absent for one day each, and so on. We arrange the given information in a cumulative frequency table, as follows:

Worked example 1 (continued)

Note how the cumulative frequencies are calculated. Also note that the last cumulative frequency (52) is the same as the total number of data values.

Days absent	Frequency	Cumulative frequency
0	2	2
1	3	5 ($2 + 3 = 5$)
2	6	11 ($5 + 6 = 11$)
3	7	18 ($11 + 7 = 18$)
4	6	24 ($18 + 6 = 24$)
5	9	33 ($24 + 9 = 33$)
6	11	44 ($33 + 11 = 44$)
7	6	50 ($44 + 6 = 50$)
8	0	50 ($50 + 0 = 50$)
9	2	52 ($50 + 2 = 52$)

In the previous worked example, we used the original data values in the frequency table. This is an example of **ungrouped data**. In the next worked example, we will group the data into **class intervals**. This is an example of **grouped data**.

New words

ungrouped data: data values that are used individually

grouped data: data values that are grouped into class intervals

class interval: an interval with a lower and upper value within which data is grouped

Worked example 2

The following marks out of 100 were obtained by 60 learners in a Mathematics examination.

a) Construct a frequency table for this data.

Group the data in these class intervals:
50–54, 55–59 and so on.

b) Use your frequency table in Question a)

to construct a cumulative frequency table for this data.

75 50 52 70 73 81 82 62 63 66
90 58 76 88 90 67 70 57 65 68
59 61 69 71 55 72 72 74 73 84
70 77 74 62 71 68 72 78 78 69
79 64 76 80 75 77 82 79 76 74
78 85 80 81 88 82 80 82 89 86

Answers

Marks	Tally	Frequency
50-54		2
55-59		4
60-64		5
65-69		7
70-74		13
75-79		12
80-84		10
85-89		5
90-94		2

Marks	Frequency	Cumulative frequency
50-54	2	2
55-59	4	6
60-64	5	11
65-69	7	18
70-74	13	31
75-79	12	43
80-84	10	53
85-89	5	58
90-94	2	60

Activity 1

1 A group of 20-year olds who use condoms regularly were asked how many sexual partners they have already had. The results are shown alongside.

3	1	4	5	5	3	5	2	2	3
4	1	1	6	4	1	5	2	2	2
4	2	5	6	1	3	5	3	3	2
5	4	6	6	1	2	3	3	4	3
4	6	4	5	1	2	2	5	2	2

a) Construct a frequency table for this data. Do not group the data.

b) Use your frequency table in Question a) to construct a cumulative frequency table for this data.

2 The same group of 20-year olds were asked at what age they started using condoms.

The results are shown alongside.

17	14	18	17	13	19	16	14	16	19
18	17	19	18	14	18	14	15	16	18
13	17	18	16	19	18	16	17	17	18
15	15	19	18	19	17	19	18	17	18
19	17	20	16	20	19	16	15	20	15

a) Construct a frequency table for this data. Group the data in these class intervals: 13-14, 15-16, and so on.

b) Use your frequency table in Question a) to construct a cumulative frequency table for this data.

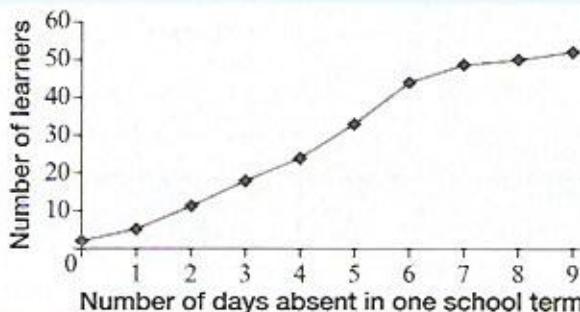
Draw cumulative frequency curves

A cumulative frequency curve or ogive is a curve obtained by plotting a graph of the cumulative frequencies against the number of data values.

Worked example 3

Draw a cumulative frequency curve for the data in Worked example 1.

Answer



Activity 2

1 Draw a cumulative frequency curve for the data in Worked example 2.

2 A group of 30 learners were asked to recite the 10 basic road rules for driving. The following were the numbers of rules correctly recited by each learner.

1	0	3	4	8	10	7	4	3	1
5	6	5	4	3	5	3	6	5	1
5	4	6	4	2	5	7	7	9	5

Activity 2 (continued)

a) Draw up and complete a cumulative frequency table with these headings:
Number of road rules / Frequency / Cumulative frequency.
b) Use your cumulative frequency table in Question a) to draw a cumulative frequency curve of this data.

3 The table shows the

marks out of 30

obtained by learners
in a Social Sciences test.

Marks	1-5	6-10	11-15	16-20	21-25	26-30
Frequency	2	6	12	16	8	4

a) Find the total number of pupils who wrote the test.
b) Draw up and complete a table with these headings: Marks / Frequency / Cumulative frequency.
c) Use your cumulative frequency table in Question b) to draw a cumulative frequency curve of this data.

Draw relative cumulative frequency curves

A **relative cumulative frequency curve** is similar to a cumulative frequency curve, but instead of plotting cumulative frequencies, we plot relative cumulative frequencies. A **relative cumulative frequency** is a cumulative frequency divided by the total frequency.

Worked example 4

Draw a relative cumulative frequency curve for the data in Worked example 1.

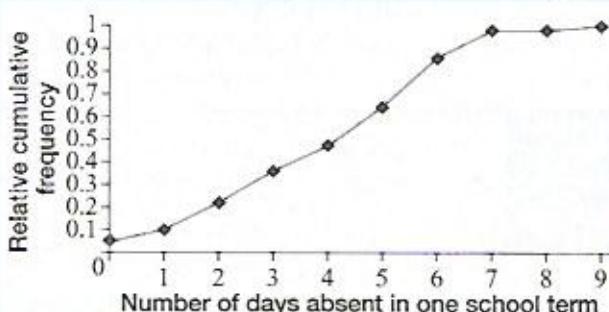
Answer

First, we calculate the relative cumulative frequencies. We use the table for cumulative frequency from Worked Example 1, then we add a column for relative cumulative frequencies.

Days absent	Cumulative frequency	Relative cumulative frequency
0	2	$\frac{2}{52} = 0.04$
1	5	$\frac{5}{52} = 0.10$
2	11	$\frac{11}{52} = 0.21$
3	18	$\frac{18}{52} = 0.35$
4	24	$\frac{24}{52} = 0.46$
5	33	$\frac{33}{52} = 0.63$
6	44	$\frac{44}{52} = 0.85$
7	50	$\frac{50}{52} = 0.96$
8	50	$\frac{50}{52} = 0.96$
9	52	$\frac{52}{52} = 1$

Worked example 4 (continued)

Now we can plot the relative cumulative frequency curve, as follows.



Note the following:

- In a cumulative frequency curve, we plot actual numbers of learners.
- In a relative cumulative frequency curve, we plot percentages of the total number of learners as decimals.
- The vertical axis of a relative cumulative frequency curve is numbered from 0 to 1.

Activity 3

1 Refer back to the data in Worked example 2.

- Copy and complete the table alongside.
- Draw a relative cumulative frequency curve for the data in the table in Question a).

Marks	Cumulative frequency	Relative cumulative frequency
50-54	2	$\frac{2}{60} = 0.03$
55-59	6	$\frac{6}{60} = 0.1$
60-64	11	
65-69	18	
70-74	31	
75-79	43	
80-84	53	
85-89	58	
90-94	60	

2 Refer back to Question 2 of Activity 2.

- Draw up and complete a table with these headings: Number of road rules / Cumulative frequency / Relative cumulative frequency.
- Draw a relative cumulative frequency curve for the data in the table in Question a).

3 Refer back to Question 3 of Activity 2.

- Draw up and complete a table with these headings: Marks / Cumulative frequency / Relative cumulative frequency.
- Draw a relative cumulative frequency curve for the data in the table in Question a).

SUB-TOPIC 2 Measures of dispersion

Introduction

Measures of dispersion give us information about how spread out or close together the data values in a data set are. There are different measures of dispersion. In this sub-topic, you will learn about the range, the quartiles, the interquartile range, the semi-interquartile range and the percentiles.

Calculate the range, interquartile range, and the semi-interquartile range

The range

The range of a data set is the difference between highest and the lowest data values in a data set. So, the range = highest data value – lowest data value.

New words

measures of dispersion: statistical measures that give us information about how spread out or close together the data values in a data set are

range: the difference between highest and the lowest data values in a data set

Worked example 5

Find the range of the following data set: {11, 9, 15, 18, 10, 12, 8}.

Answer

First order the data values from lowest to highest: 8, 9, 10, 11, 12, 15, 18.

The range = highest data value – lowest data value = $18 - 8 = 10$.

Activity 4

1 Find the range of each of the following data sets.

- {17, 18, 5, 25, 1, 16, 2, 8, 20}
- {7.8, 1.8, 8.9, 2.1, 4.5, 4.1, 4.7, 5.7, 9.8, 4.8}
- {-19 °C, 0 °C, 31 °C, -15 °C, -2 °C}

2 Look at the frequency table below.

Marks	1-5	6-10	11-15	16-20	21-25	26-30
Frequency	2	6	12	16	8	4

- What is the largest possible range of this data? Explain your reasoning.
- What is the smallest possible range of this data? Explain your reasoning.

3 Given the following data set: {49, 29, 37, 33, 48, 38, 40, x , 48, 27, 50}. If this data set has a range of 31, find two possible values for x . Explain your reasoning.

The quartiles

The quartiles divide an ordered data set into four equal quarters. There are three quartiles: the lower quartile, the middle quartile and the upper quartile.

- The **middle quartile** (Q_2) divides the data set into two equal halves. It is the same as the median.
- The **lower quartile** (Q_1) divides the bottom half of the data set into two equal halves.
- The **upper quartile** (Q_3) divides the top half of the data set into two equal halves.

New words

quartiles: measures of dispersion that divide an ordered data set into four equal quarters

middle quartile: a measure of dispersion that divides an ordered data set into two equal halves

lower quartile: a measure of dispersion that divides the bottom half of an ordered data set into two equal halves

upper quartile: a measure of dispersion that divides the top half of an ordered data set into two equal halves

Worked example 6

1 The daily maximum temperatures in Lusaka during July one year were as follows:

22 °C	21.5 °C	21 °C	22 °C	21.5 °C	22.5 °C	22 °C	21.5 °C	20 °C	19.5 °C
19 °C	20 °C	21.5 °C	22 °C	22.5 °C	23.5 °C	23 °C	22.5 °C	23 °C	23.5 °C
23.5 °C	24 °C	26.5 °C	27 °C	24.5 °C	23.5 °C	24 °C	24.5 °C	24 °C	23 °C
24.5 °C									

Calculate the value of:

a) Q_2 b) Q_1 c) Q_3

2 The heights of ten friends were recorded as follows:

176 cm	180 cm	159 cm	152 cm	168 cm
156 cm	166 cm	151 cm	168 cm	161 cm

Calculate the value of:

a) Q_2 b) Q_1 c) Q_3

Answers

1 First write these temperatures in ascending order, as follows:

19 °C, 19.5 °C, 20 °C, 20 °C, 21 °C, 21.5 °C, 21.5 °C, 21.5 °C, 22 °C, 22 °C, 22 °C, 22 °C, 22.5 °C, 22.5 °C, 22.5 °C, 23 °C, 23 °C, 23 °C, 23.5 °C, 23.5 °C, 23.5 °C, 24 °C, 24 °C, 24 °C, 24.5 °C, 24.5 °C, 24.5 °C, 26.5 °C, 27 °C

a) There are 31 data values.

The median (Q_2) is the middle value of the data set. This is the 16th data value.

So, $Q_2 = 22.5$ °C.

Worked example 6 (continued)

b) The bottom half of the data set consists of the first 15 data values.
 (Note that the median does not form part of the bottom half of the data set.)
 Q_1 is the middle value of the bottom half of the data set. This is the 8th data value.
 $So, Q_1 = 21.5^\circ C.$

c) The top half of the data set consists of the last 15 data values.
 (Again, we ignore the median.)
 Q_3 is the middle value of the top half of the data set. This is the 24th data value.
 $So, Q_3 = 24^\circ C.$

2 First write these heights in ascending order, as follows:
 151 cm, 152 cm, 156 cm, 159 cm, 161 cm, 166 cm, 168 cm, 168 cm, 176 cm, 180 cm

a) There are 10 data values.
 The median (Q_2) is the middle value of the data set. This is halfway between the 5th and the 6th data values.
 $So, Q_2 = \frac{(161 + 166)}{2} = 163.5 \text{ cm.}$

b) The bottom half of the data set consists of the first 5 data values.
 Q_1 is the middle value of the bottom half of the data set. This is the 3rd data value.
 $So, Q_1 = 156 \text{ cm.}$

c) The top half of the data set consists of the last 5 data values.
 Q_3 is the middle value of the top half of the data set. This is the 8th data value.
 $So, Q_3 = 168 \text{ cm.}$

Activity 5

For each of the following data sets, calculate the value of

a) Q_2
 b) Q_1
 c) Q_3 .

1 $\{25, 25, 31, 15, 32, 14, 20, 21, 31, 32\}$
 2 $\{-4, 16, -6, -14, 14, -14, -19, 7, -3, 17, 15, -13\}$
 3 $\{196, 337, 731, 625, 799, 565, 186, 736, 496\}$
 4 $\{6.4, 1.9, 2.6, 3.7, 8.8, 5, 6.3, 8\}$
 5 $\{1.8, 0.6, 1.9, -2, -1.7, -0.1, 1.6, -1.8, -1, -1.6, -1.2, 2.1, -0.5, 1.4, -2.6\}$

The interquartile range and the semi-interquartile range

The **interquartile range** (IQR) is the difference between the upper quartile (Q_3) and the lower quartile (Q_1).

$$\text{So, IQR} = Q_3 - Q_1.$$

The **semi-interquartile range** is half the interquartile range.

$$\text{So, the semi-interquartile range} = \frac{1}{2}(Q_3 - Q_1).$$

New words

interquartile range: the difference between the upper and lower quartiles of a data set
semi-interquartile range: half the interquartile range

Worked example 7

For the data set {10, 17, 9, 15, 12, 11, 20}, find:

- the lower quartile (Q_1)
- the upper quartile (Q_3)
- the interquartile range
- the semi-interquartile range.

Answers

First sort the data values in ascending order: {9, 10, 11, 12, 15, 17, 20}.

- There are seven numbers in the data set. $Q_1 = 10$.
- $Q_3 = 17$.
- The interquartile range $= Q_3 - Q_1 = 17 - 10 = 7$.
- The semi-interquartile range $= \frac{1}{2}(Q_3 - Q_1) = \frac{1}{2} \times 7 = 3.5$.

Activity 6

For each of the data sets in Activity 5, calculate the interquartile range and the semi-interquartile range.

Thus far, you have worked with the quartiles of ungrouped data. You will now learn how to calculate the quartiles of grouped data.

Worked example 8

Over a 2-year period, 600 women were treated for cervical cancer in a hospital in Zambia.

Some data recorded about their treatment is summarised alongside.

- Calculate the mean mass of tablets taken by these patients.

Mass of tablets consumed (g)	Number of women
$120 < x \leq 130$	20
$130 < x \leq 140$	85
$140 < x \leq 150$	320
$150 < x \leq 160$	140
$160 < x \leq 170$	35

Worked example 8 (continued)

b) Copy and complete the cumulative frequency table below.

Mass of tablets consumed (g)	Frequency	Cumulative frequency
$x \leq 120$	0	0
$120 < x \leq 130$	20	20
$130 < x \leq 140$	85	105
$140 < x \leq 150$	320	
$150 < x \leq 160$	140	
$160 < x \leq 170$	35	

- c) Use suitable scales on both axes to draw a cumulative frequency curve of this data.
- d) Use your curve to estimate:
 - (i) the median
 - (ii) the lower quartile
 - (iii) the upper quartile
 - (iv) the interquartile range
 - (v) the semi-interquartile range.
- e) What does the lower quartile tell us in this situation?
- f) What does the median tell us in this situation?

Answers

a) To calculate the mean mass, we first find the midpoints of each class interval, as shown below.

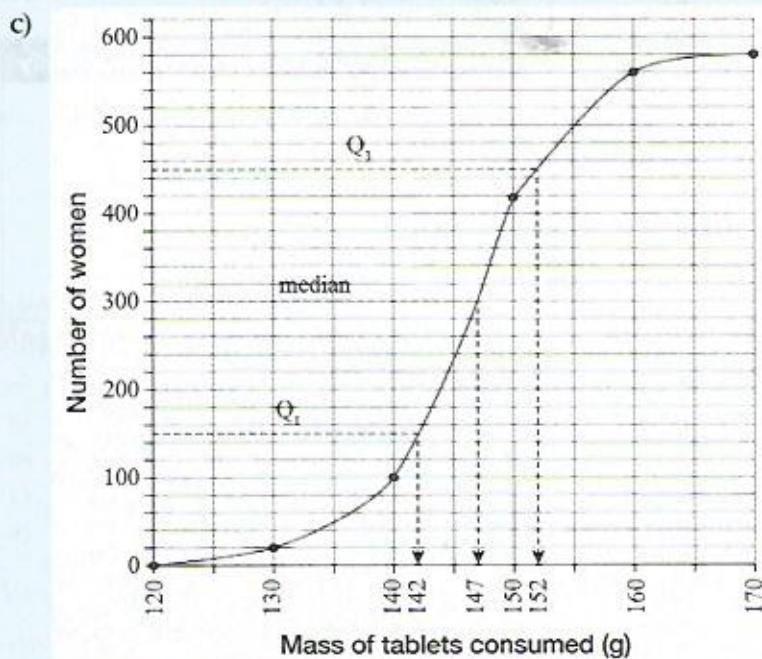
Mass of tablets consumed (g)	$120 < x \leq 130$	$130 < x \leq 140$	$140 < x \leq 150$	$150 < x \leq 160$	$160 < x \leq 170$
Frequency (f)	20	85	320	140	35
Midpoint (x)	125	135	145	155	165

To calculate the mean, for which we use the symbol \bar{x} , we multiply each midpoint by its frequency, add all of those products together, and then divide the answer by the total of all the frequencies.

$$\begin{aligned}
 \text{So, } \bar{x} &= \frac{\sum f x}{\sum f}, \text{ where } f \text{ is each frequency and } x \text{ is each midpoint} \\
 &= \frac{(20 \times 125) + (85 \times 135) + (320 \times 145) + (140 \times 155) + (35 \times 165)}{20 + 85 + 320 + 140 + 35} \\
 &= \frac{87\ 850}{600} = 146.42 \text{ g}
 \end{aligned}$$

b)	Mass of tablets consumed (g)	Frequency	Cumulative frequency
	$x \leq 120$	0	0
	$120 < x \leq 130$	20	20
	$130 < x \leq 140$	85	105
	$140 < x \leq 150$	320	425
	$150 < x \leq 160$	140	565
	$160 < x \leq 170$	35	600

Worked example 8 (continued)



d) (i) The median corresponds to $\frac{1}{2}$ of the total frequency.
 $\frac{1}{2} \times 600 = 300$. The value of 300 on the vertical axis corresponds to the value of 147 g on the horizontal axis, as shown on the graph by dotted lines.
So, the median = 147 g.

(ii) The lower quartile corresponds to $\frac{1}{4}$ of the total frequency.
 $\frac{1}{4} \times 600 = 150$. The value of 150 on the vertical axis corresponds to the value of 142 g on the horizontal axis, as shown on the graph by dotted lines.
So, the lower quartile = 142 g.

(iii) The upper quartile corresponds to $\frac{3}{4}$ of the total frequency.
 $\frac{3}{4} \times 600 = 450$. The value of 450 on the vertical axis corresponds to the value of 152 g on the horizontal axis, as shown on the graph by dotted lines.
So, the upper quartile = 152 g.

(iv) The interquartile range = $Q_3 - Q_1 = 152 \text{ g} - 142 \text{ g} = 10 \text{ g}$.

(v) The semi-interquartile range = $\frac{1}{2}(Q_3 - Q_1) = \frac{1}{2}(10 \text{ g}) = 5 \text{ g}$.

e) The lower quartile tells us that $\frac{1}{4}$ (or 25%) of the women consumed less than 142 g of the tablets each.

f) The median tells us that $\frac{1}{2}$ (or 50%) of the women consumed less than 147 g of the tablets each.

Activity 7

1 800 learners in a school sold raffle tickets to fund an AIDS-awareness campaign in their community. Their sales are summarised in the table below.

Number of tickets sold	1-20	21-40	41-60	61-80	81-100	101-120	121-140
Number of learners	125	174	136	119	98	83	65

a) Draw up a cumulative frequency table for this data.
 b) Draw a cumulative frequency curve of this data.
 c) Use your graph to estimate:
 (i) the middle quartile
 (ii) the lower quartile
 (iii) the upper quartile
 (iv) the interquartile range
 (v) the semi-interquartile range.

2 The time that it took people to vote at a polling station in a recent by-election was as follows.

a) How many people voted altogether?
 b) Calculate the mean time that they took to vote.



Time (minutes)	Number of voters
$0 < x \leq 20$	146
$20 < x \leq 40$	404
$40 < x \leq 60$	762
$60 < x \leq 80$	291
$80 < x \leq 100$	115
$100 < x \leq 120$	82

c) Copy and complete the cumulative frequency table below.

Time (minutes)	Frequency	Cumulative frequency
$0 < x \leq 20$	146	
$20 < x \leq 40$	404	
$40 < x \leq 60$	762	
$60 < x \leq 80$	291	
$80 < x \leq 100$	115	
$100 < x \leq 120$	82	

d) Use suitable scales on both axes to draw a cumulative frequency curve of this data.
 e) Use your curve to estimate:
 (i) the median
 (ii) the lower quartile
 (iii) the upper quartile
 (iv) the interquartile range
 (v) the semi-interquartile range.



Did you know?

Every Zambian citizen who has the right to vote should use this right at every opportunity. So when you are old enough to vote, remember to register as a voter and vote whenever possible. In this way, you can play a part in electing your country's government.

Calculate the percentiles

While quartiles divide an ordered data set into four equal quarters, percentiles divide an ordered data set into 100 equal hundredths. The n th percentile of a data set is that value such that $n\%$ of the data values are less than or equal to that value.

Note that the 25th percentile corresponds with the lower quartile, the 50th percentile corresponds with the middle quartile (the median) and the 75th percentile corresponds with the upper quartile.

New word

percentiles: measures of dispersion that divide an ordered data set into 100 equal hundredths

Worked example 9

One way to determine whether or not a person is over- or underweight is to use a calculation based on their mass and height. This is called a Body Mass Index (BMI) and is calculated as follows: $\text{BMI} = \frac{\text{mass in kilograms}}{\text{height in metres}^2}$.

The BMI of an individual is interpreted as follows:

BMI	Category
$x < 15$	Very severely underweight
$15 \leq x < 16$	Severely underweight
$16 \leq x < 18.5$	Underweight
$18.5 \leq x < 25$	Normal
$25 \leq x < 30$	Overweight
$30 \leq x < 35$	Moderately obese
$35 \leq x < 40$	Severely obese
$40 \leq x$	Very severely obese

A school principal decided to launch a health-awareness campaign in her school by determining the BMI of her learners and staff. The results were as follows:

BMI	Number of learners and staff
$x < 15$	0
$15 \leq x < 16$	16
$16 \leq x < 18.5$	132
$18.5 \leq x < 25$	423
$25 \leq x < 30$	157
$30 \leq x < 35$	54
$35 \leq x < 40$	18
$40 \leq x$	0

Worked example 9 (continued)

- How many learners and staff members were measured for this campaign?
- What percentage of those learners and staff members were of normal weight?
- Draw up a cumulative frequency table of this data.
- Draw a cumulative frequency curve for this data.
- Use your curve to estimate:
 - the 10th percentile
 - the 80th percentile
 - between which two percentiles the people of normal weight fall.

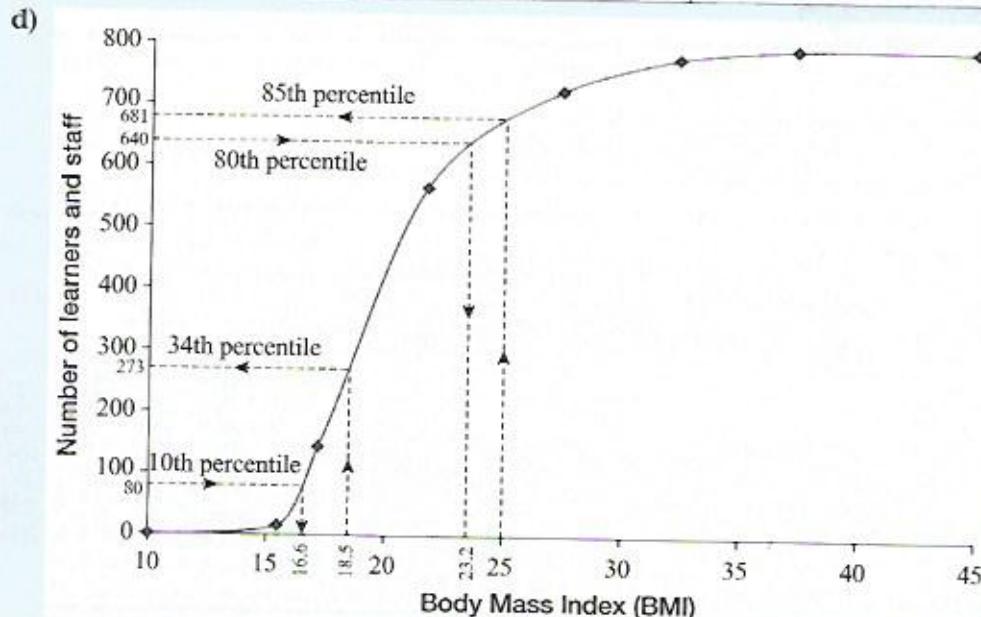
Answers

a) $0 + 16 + 132 + 423 + 157 + 54 + 18 + 0 = 800$
 So, 800 learners and staff members were measured for this campaign.

b) $\frac{423}{800} \times 100\% = 52.875\%$
 So, 52.875% of those learners and staff members were of normal weight.

c)

BMI	Frequency	Cumulative frequency
$x < 15$	0	0
$15 \leq x < 16$	16	16
$16 \leq x < 18.5$	132	148
$18.5 \leq x < 25$	423	571
$25 \leq x < 30$	157	728
$30 \leq x < 35$	54	782
$35 \leq x < 40$	18	800
$40 \leq x$	0	800



Worked example 9 (continued)

e) (i) The 10th percentile corresponds to 10% of the total frequency.
 $\frac{10}{100} \times 800 = 80$. The value of 80 on the vertical axis corresponds to the value of 16.6 on the horizontal axis, as shown on the graph by dotted lines.
So, the 10th percentile gives a BMI of 16.6.

(ii) The 80th percentile corresponds to 80% of the total frequency.
 $\frac{80}{100} \times 800 = 640$. The value of 640 on the vertical axis corresponds to the value of 23.2 on the horizontal axis, as shown on the graph by dotted lines.
So, the 80th percentile gives a BMI of 23.2.

(iii) The people of normal weight have a BMI of between 18.5 and 25.
The value of 18.5 on the horizontal axis corresponds to the value of 273 on the vertical axis, as shown on the graph by dotted lines.
 $\frac{273}{800} \times 100\% \approx 34\%$, so this is the 34th percentile.
The value of 25 on the horizontal axis corresponds to the value of 681 on the vertical axis, as shown on the graph by dotted lines.
 $\frac{681}{800} \times 100\% \approx 85\%$, so this is the 85th percentile.
So, the people of normal weight fall between the 34th and the 85th percentiles.
(Notice that the difference between these two percentiles = $85 - 34 = 51$, which is very close to the answer of 52.875% in Question b). This means that our estimates from the graph are reasonably accurate!)

Activity 8

- 1 Refer back to the previous example. Use the cumulative frequency curve to estimate:
 - a) the 50th percentile
 - b) the percentile in which a learner with a BMI of 30 will fall.
- 2 The distances that learners in a school travel every morning to get to school are as follows:

Distance (kilometres)	Number of learners
$0 < x < 1$	286
$1 \leq x < 2$	348
$2 \leq x < 3$	195
$3 \leq x < 4$	160
$4 \leq x < 5$	92
$5 \leq x < 6$	63
$6 \leq x < 7$	42
$7 \leq x < 8$	14



Activity 8 (continued)

- How many learners are there in the school?
- Draw up a cumulative frequency table of this data.
- Draw a cumulative frequency curve for this data.
- Use your curve to estimate:
 - the 20th percentile
 - the 40th percentile
 - the 60th percentile
 - the 80th percentile.

Calculate variance and standard deviation

The final two measures of dispersion that we will look at are the variance and the standard deviation. These measures give us information about how far the data values in a data set are from the mean of the data set.

The **variance** is a measure of the average distance between each number in a data set and the mean. We calculate the variance of a data set as follows:

variance (σ^2) = $\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n}$ for ungrouped data,
where x is each score, \bar{x} is the mean and n is the total frequency.

variance (σ^2) = $\frac{\sum_{i=1}^n f_i(x_i - \bar{x})^2}{n}$ for grouped data, where f is the frequency of each class interval, x is the midpoint of each class interval, \bar{x} is the mean and n is the total frequency.

New words

variance: a measure of the average distance between each number in a data set and the mean

standard deviation: the square root of the variance

The **standard deviation** is equal to the square root of the variance. So, standard deviation = $\sqrt{\sigma^2}$.

Note the following important points about variance and standard deviation:

- The smaller the variation and the standard deviation, the closer the data values are to the mean. So, the data values are close together.
- The larger the variation and the standard deviation, the further the data values are from the mean. So, the data values are spread out.

Worked example 10

The following table shows the distribution of lengths of residential plots in a village.

Length of plots (m)	6-10	11-15	16-20	21-25	26-30	31-35
Frequency	2	13	47	33	10	6

For the data above, calculate:

- the mean
- the variance
- the standard deviation

Worked example 10 (continued)

Answers

a) To calculate the mean, we must find the midpoint of each class interval and then calculate fx for each class interval, as shown in the table below.

Length (m)	Frequency (f)	Midpoint (x)	fx
6-10	2	8	16
11-15	13	13	169
16-20	47	18	846
21-25	33	23	759
26-30	10	28	280
31-35	6	33	198
Totals	111		2 268

So, $\bar{x} = \frac{\sum fx}{\sum x}$, where f is each frequency and x is each midpoint

$$\begin{aligned}
 &= \frac{(16 + 169 + 846 + 759 + 280 + 198)}{111} \\
 &= \frac{2 268}{111} \\
 &= 20.43
 \end{aligned}$$

b) Now we need to add three more columns to the table for the following: $(x - \bar{x})$, $(x - \bar{x})^2$ and $f(x - \bar{x})^2$, as shown below.

Length (m)	Frequency (f)	Midpoint (x)	fx	$x - \bar{x}$	$(x - \bar{x})^2$	$f(x - \bar{x})^2$
6-10	2	8	16	-12.43	154.50	309.00
11-15	13	13	169	-7.43	55.20	717.60
16-20	47	18	846	-2.43	5.90	277.30
21-25	33	23	759	2.57	6.60	217.80
26-30	10	28	280	7.57	57.30	573.00
31-35	6	33	198	12.57	158.00	948.00
Totals	111		2 268			3 042.70

$$\begin{aligned}
 \text{Variance } (\sigma^2) &= \sum_{i=1}^n \frac{f(x_i - \bar{x})^2}{n} \\
 &= \frac{3 042.7}{111} \\
 &= 27.41
 \end{aligned}$$

c) Standard deviation (σ) = $\sqrt{\sigma^2} = \sqrt{27.41} = 5.24$

Activity 9

1 For each of the following data sets, calculate:

- the mean
- the variance
- the standard deviation.

a) {1, 3, 5, 7, 10} b) {-10 °C, 0 °C, 30 °C, 10 °C, 20 °C}
 c) {19, 22, 25, 21, 23, 22} d) {0.2, 0.5, 0.9, 1.2, 0.3, 0.8}

2 The table below shows the masses of the learners in a class.

Mass (kg)	50-54	55-59	60-64	65-69	70-74	75-79	80-84
Frequency	3	4	6	8	7	5	2

a) How many learners are there in the class?

b) Calculate:

- the mean
- the variance
- the standard deviation of this data.

3 Compare the data sets below:

A: {1, 2, 3, 4, 5} B: {1, 4, 9, 16, 25} C: {1, 8, 27, 64, 125}

a) Which data set would you expect to have the smallest standard deviation? Give a reason for your answer.

b) Calculate the standard deviation for each of the three sets. Do your answers support your prediction in Question a)?

4 The police force in a community recorded the number of drug-related crimes that were reported weekly on a number of consecutive weeks. The table below shows the results.

Number of drug-related crimes	1-5	6-10	11-15	16-20	21-25
Number of weeks	4	8	13	10	17

a) Over what period of time was this data collected?

b) What is the maximum range of this data?

c) Calculate:

- the mean
- the variance
- the standard deviation of this data.



Summary

Cumulative frequency tables

- A cumulative frequency table is a table that has three columns: one each for data values, frequencies and cumulative frequencies.
- The frequency of a data value is the number of times that the data value appears in a data set.
- The cumulative frequency of a data value is the sum of all the frequencies up to and including the frequency of that data value.
- Ungrouped data are data values that are used individually.
- Grouped data are data values that are grouped into class intervals.
- A class interval is an interval with a lower and upper value within which data is grouped.
- A cumulative frequency curve or ogive is a curve obtained by plotting a graph of the cumulative frequencies against the number of data values.
- A relative cumulative frequency curve is similar to a cumulative frequency curve, but instead of plotting cumulative frequencies, we plot relative cumulative frequencies. A relative cumulative frequency is a cumulative frequency divided by the total frequency.
- The vertical axis of a relative cumulative frequency curve is numbered from 0 to 1.

Measures of dispersion

- Measures of dispersion give us information about how spread out or close together the data values in a data set are. These include the range, the quartiles, the interquartile range, the semi-interquartile range and the percentiles.
- The range of a data set is the difference between highest and the lowest data values in a data set.
- The quartiles divide an ordered data set into four equal quarters. There are three quartiles: the lower quartile, the middle quartile and the upper quartile.
- The middle quartile (Q_2) divides the data set into two equal halves. It is the same as the median.
- The lower quartile (Q_1) divides the bottom half of the data set into two equal halves.
- The upper quartile (Q_3) divides the top half of the data set into two equal halves.
- The **interquartile range (IQR)** is the difference between the upper quartile (Q_3) and the lower quartile (Q_1).
- The **semi-interquartile range** is half the interquartile range.
- The **percentiles** divide an ordered data set into 100 equal hundredths.
- The n th percentile of a data set is that value such that $n\%$ of the data values are less than or equal to that value.

- The **variance** is a measure of the average distance between each number in a data set and the mean.
- The **standard deviation** is equal to the square root of the variance.

Revision exercises (remedial)

1 The table alongside shows the number of siblings (brothers and sisters) of the learners in a Grade 11 class.

a) What do we call this kind of table? (1)
 b) Copy and complete the table. (7)

2 Calculate the range of the following data set:
 $\{13, 19, 27, 3, 17, 39, 6, 17\}$. (2)

Number of Siblings	Tally	Frequency	Cumulative frequency
0			
1			
2			
3			
4			
5			
6			

Revision exercises

3 An English teacher challenged her class to read at least four English books for their own pleasure during the coming month. At the end of the month, she asked her class how many books they had read. The results were as follows:

1	0	3	1	8	5	1	0	4	2
8	3	3	6	6	8	0	0	1	7
3	0	2	2	6	5	6	3	2	0
2	0	5	4	4	5	6	3	4	8

a) Construct a cumulative frequency table for this data. Do not group the data. (9)
 b) Draw a cumulative frequency curve of this data. (7)

4 For the data set $\{16, 6, 5, 2, 10, 21, 24, 8, 14, 10, 19, 20\}$, find:

a) the range (2)
 b) the median (Q_2) (2)
 c) the lower quartile (Q_1) (2)
 d) the upper quartile (Q_3) (2)
 e) the interquartile range (2)
 f) the semi-interquartile range. (2)

Revision and assessment continued

5 For the data set {5, 4, 5, 9, 7, 6}, calculate:

- a) the mean (2)
- b) the variance (8)
- c) the standard deviation. (2)

Total marks: 50

Assessment exercises

1 A school principal launched a campaign to encourage decent behaviour amongst her learners. For positive actions that promoted gender equality, human rights, awareness of the environment and of people with special needs, she awarded certificates for good behaviour to the deserving learners once a week. During 40 school weeks, she awarded the following numbers of certificates.



Certificates awarded weekly	1-5	6-10	11-15	16-20	21-25	26-30
Number of weeks	12	10	8	5	4	1

- a) Draw up a cumulative frequency table for this data. (9)
- b) Use suitable scales on both axes to draw a cumulative frequency curve of this data. (7)
- c) Use your cumulative frequency curve to estimate:
 - (i) the median (3)
 - (ii) the lower quartile (3)
 - (iii) the upper quartile (3)
 - (iv) the interquartile range (1)
 - (v) the 15th percentile (3)
 - (vi) the 85th percentile. (3)
- d) Use calculations to estimate:
 - (i) the mean (4)
 - (ii) the variance (10)
 - (iii) the standard deviation. (2)
- e) Explain why your answers to Question 1d are estimates and not exact. (2)

2 Consider the data set $(x, x + 2, x + 4, \dots, x + 18)$, where x is any real number.

- a) Use algebraic techniques to calculate the variance of this data set. (8)
- b) Hence, calculate the standard deviation of this data set. (2)

Total marks: 60

Glossary

A

absolute error: the absolute difference between the true value and the recorded value of a dimension 4

absolute value: the numeric value of a number without its sign 20

addition rule for mutually exclusive events: $P(A \text{ or } B) = P(A) + P(B)$ 153

ambiguous case: when the sine rule has two solutions for a given triangle 121

amplitude: half the distance between the highest and lowest points of a wave 128

AP: the abbreviation of the term "arithmetic progression" 12

arc: part of the circumference of a circle 85

area rule: the rule to calculate the area of any triangle, when two sides and the included angle are given 125

area: the amount of space that a two-dimensional figure occupies 136

arithmetic mean: the middle term of three consecutive terms in an AP 14

arithmetic progression: a sequence in which each term is formed by adding a constant amount to the previous term 12

arithmetic sequence: another name for an arithmetic progression 12

arithmetic series: the sum of an arithmetic progression 15

asymptote: a straight line to which a graph draws closer and closer, without ever touching it 128

axis of symmetry: the vertical line that passes through the turning point of a parabola 52

B

bisect: divide into two equal halves 106

C

CAST diagram: a visual aid used to determine the signs of the sin, cos and tan ratios in the different quadrants of the Cartesian plane 129

chord: a straight line drawn across a circle that joins any two points on its circumference 85

class interval: an interval with a lower and upper value within which data is grouped 161

common difference: the constant difference between any two consecutive terms in an arithmetic progression 12

common ratio: the constant ratio between any two consecutive terms in a geometric progression 17

composite function: a function that is a combination of two or more functions 45

compound events: two or more events that together form one trial of an experiment 151

cone: a three-dimensional object that has a circular base and one vertex 140

constant of variation: the constant, k , in a variation equation 75

continuous sample space: a sample space containing an infinite number of outcomes 156

convergent GP: a GP of which the terms approach a limit (consecutive terms become closer together) 22

cosine rule: a rule to solve non right-angled triangles, when two sides and the included angle are given, or three sides are given 123

cumulative frequency (of a data value): the sum of all the frequencies up to and including the frequency of that data value 160

cumulative frequency table: a table that has columns for data values, frequencies and cumulative frequencies 160

cyclic quadrilateral: a quadrilateral of which all four vertices lie on the circumference of the same circle 90

D

diameter: a chord that passes through the centre of a circle 85

dimension: a measurement 2

direct variation: a relationship between two variables, such that as one variable increases, the other increases 75
discrete sample space: contains a finite number of outcomes 156
divergent GP: a GP of which the terms do not approach a limit (consecutive terms become further apart) 22
domain: the set of x -values for which a function is defined 42

E

equidistant (from): equally far from, the same distance from 107
event: one particular outcome in which we are interested 148

F

frequency (of a data value): the number of times that the data value appears in a data set 160
frustum: part of a cone or pyramid that remains if the top has been cut off 142
function: a one-to-one or a many-to-one mapping 42

G

geometric mean: the middle term of three consecutive terms in a GP 19
geometric progression: a sequence in which each term is formed by multiplying the previous term by a constant amount 17
geometric sequence: another name for a geometric progression 17
geometric series: the sum of a geometric progression 20
GP: the abbreviation of the term “geometric progression” 17
gradient (of a line segment): the steepness, or slope of the line 30
grouped data: data values that are grouped into class intervals 161

I

independent events: events that cannot influence one another 154
infinite progression: a progression that has an infinite number of terms 21

interquartile range: the difference between the upper and lower quartiles of a data set 168
inverse function: a function that reverses the mapping of the original function 42
inverse variation: a relationship between two variables, such that as one variable increases, the other decreases 76

J

joint variation: a variation where a quantity varies directly with a product of two or more other quantities 77

L

linear function: a function of the form $f(x) = ax + b$ 47
loci: the plural of locus 107
locus: the path along which a point may move in order to satisfy one or more conditions 107
lower bound: the smallest number of the interval within which a dimension can fall 2
lower limit: another name for a lower bound 2
lower quartile: a measure of dispersion that divides the bottom half of an ordered data set into two equal halves 166

M

major arc: the larger arc formed by a chord that is not a diameter 85
major sector: the larger sector formed by two radii that are not part of the same diameter 85
major segment: the larger segment formed by a chord that is not a diameter 85
mapping: the matching of the elements of one set to the elements of another set by means of a rule 42
measures of dispersion: statistical measures that give us information about how spread out or close together the data values in a data set are 165
middle quartile: a measure of dispersion that divides an ordered data set into two equal halves 166

midpoint (of a line segment): the point on the line segment that lies exactly halfway between the two endpoints of the line segment 28

minor arc: the smaller arc formed by a chord that is not a diameter 85

minor sector: the smaller sector formed by two radii that are not part of the same diameter 85

minor segment: the smaller segment formed by a chord that is not a diameter 85

multiplication rule for independent events: $P(A \text{ and } B) = P(A) \times P(B)$ 154

mutually exclusive events: events that cannot happen at the same time 153

O

ordered pairs: coordinate pairs of the form (x, y) 42

outcome: a result of an experiment 148

P

parabola: the graph of a quadratic function 52

partial variation: a variation between two variables, such that the graph of this variation is a straight line that does not pass through the origin 78

percentage error: the relative error, written as a percentage 4

percentiles: measures of dispersion that divide an ordered data set into 100 equal hundredths 172

period: the distance on the horizontal axis of one full cycle of a graph 128

perpendicular bisector: a line that bisects another line at an angle of 90° 106

probability grid: a table that shows all the possible outcomes of a trial that consists of two events 152

probability: likelihood 148

pyramid: a three-dimensional object that has a polygon as a base and all its other faces are triangles that meet at a common vertex 138

Q

quadratic equation: an equation of the form $ax^2 + bx + c = 0$, where a, b and c are constants and $a \neq 0$ 60

quadratic formula: the formula used to find the x -intercepts of a parabola 54

quadratic function: a function of the form $f(x) = ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$ 52

quartiles: measures of dispersion that divide an ordered data set into four equal quarters 166

R

radii: the plural of radius 85

radius: a line drawn from the centre of a circle to its circumference 85

range: the difference between highest and the lowest data values in a data set 165

range: the set of y -values for which a function is defined 42

reference angle: an angle in the first quadrant of the Cartesian plane, used to determine the solution of a trigonometric equation 129

regular tetrahedron: a triangular-based pyramid of which all four faces are identical equilateral triangles 138

relative cumulative frequency curve: similar to a cumulative frequency curve, but instead of plotting cumulative frequencies, we plot relative cumulative frequencies 163

relative error: the ratio of the absolute error to the true value of a dimension 4

roots (of an equation): the values of x that satisfy an equation 60

roots (of a quadratic equation): the x -intercepts of the graph of the equation 54

S

sample space: the total number of possible outcomes of a trial 148

secant: a line that cuts a circle in two points 94

sector: a part of a circle that is enclosed by two radii and an arc 85

segment: a part of a circle that is enclosed by a chord and an arc 85
self-inverse function: a function that is identical to its inverse 44
semi-circle: half a circle 85
semi-interquartile range: half the interquartile range 168
sequence: an ordered set of numbers, where a pattern exists 12
sine rule: a rule to solve non right-angled triangles, when two angles and any side are given, or two sides and the non-included angle are given 121
standard deviation: the square root of the variance 175
sum to infinity: the limiting value of the sum of a GP, as the number of terms tends to infinity 22

T

tangent: a line that touches a circle at one point only 94
term: a number in a sequence 12
total surface area: the sum of the areas of all the faces of a three-dimensional object 138
tree diagram: a diagram that is used to solve problems involving compound events 151
trial: one performance of an experiment 148
trigonometric ratios: the sine, cosine and tangent ratios, abbreviated to sin, cos and tan 115

trigonometry: the branch of mathematics that deals with the measurements of triangles 114
TSA: the abbreviation of "total surface area" 138
turning point: the point where a parabola turns; the minimum or maximum point of the parabola 52

U

ungrouped data: data values that are used individually 161
upper bound: the biggest number of the interval within which a dimension can fall 2
upper limit: another name for an upper bound 2
upper quartile: a measure of dispersion that divides the top half of an ordered data set into two equal halves 166

V

variance: a measure of the average distance between each number in a data set and the mean 175
volume: the amount of space that a three-dimensional object occupies 141

Z

zero-product rule: if $A \times B = 0$, then A must be zero or B must be zero 63

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