MESVTEE
APPROVED
2015

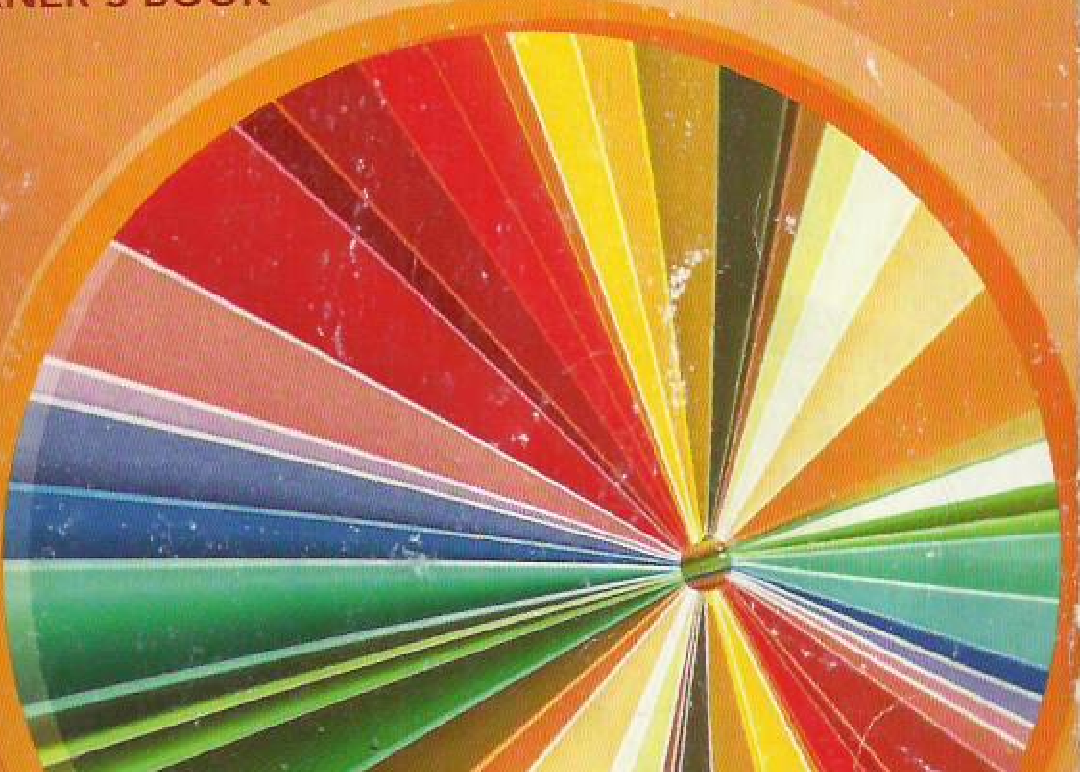
OXFORD

Progress in

12

Mathematics

LEARNER'S BOOK



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FREDERICK FINCH
SYLVIA MUKE
with **KAREN VAN NIEKERK**

OXFORD

Contents

How to use thi

Welcome to the *Progre*

This series is based

Ministry of Education,

knowledge, skills and

in Mathematics Grade 1

success in this subject.

This page will help

The book is divided

covered in your Mathe

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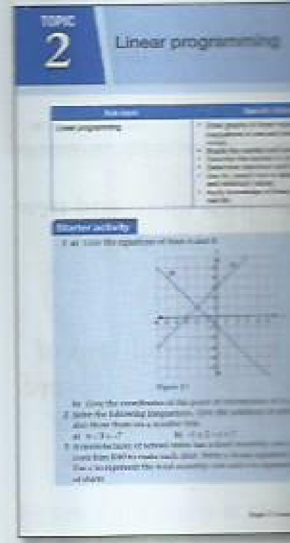

The **topic summary** will help you to revise key learning points in the topic quickly.

Revision exercises help you revise the topic's work and check your understanding.

Assessment exercises help you prepare for tests and exams.

more and new
learning.

definitions of
ain new word
initions are
f the book.

TOPIC 1

Functions

Sub-topic	Specific Outcomes
Cubic functions	• Draw graphs of cubic functions. • Use graphs to find solutions. • Determine gradients of curves. • Estimate areas under curves.
Inverse functions	• Draw graphs of inverse functions. • Application of graphs of functions.

Figure 1.1 Volume is measured in cubic units. We can find the maximum or minimum volume of containers by using cubic functions and calculus.

Starter activity

1 Given the equations of functions $f(x) = x^2$ and $g(x) = x^3$, complete the following table of values.

x	–4	–3	–2	–1	0	1	2	3	4
$f(x) = x^2$									
$g(x) = x^3$									

2 Draw graphs of these functions on the same set of axes by plotting the points and connecting them with a smooth curve.

3 Discuss with a partner the similarities and the differences of the two graphs you have drawn.

4 For both graphs, use the coordinates of the points to calculate the average gradient of the curve between $x = 1$ and $x = 2$.

SUB-TOPIC 1

Cubic functions

Determine gradients of curves

To draw the graphs of functions, we need to know which parts of graphs are increasing, decreasing and stationary (have a gradient of zero).

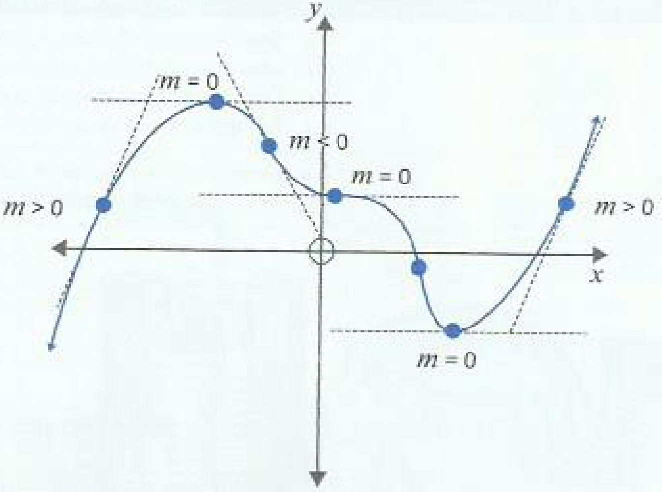

Figure 1.2

Let's revise how to find the gradient m of a line segment. The average gradient of a curve between two points is the gradient of the line segment connecting those two points. So in the graph below, the average gradient of the function f between A and B is $m = \frac{y_B - y_A}{x_B - x_A}$.

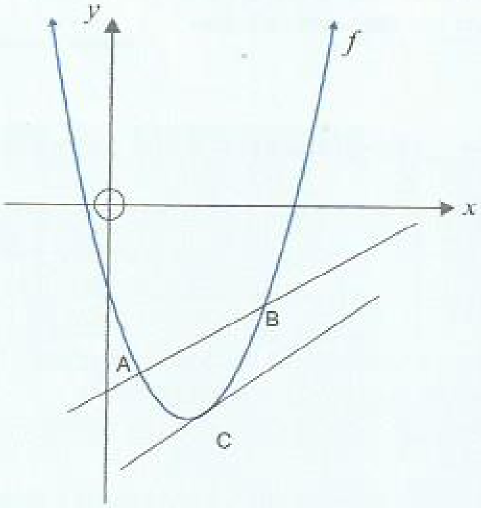

Figure 1.3

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tangent: straight line th

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of the change in the
called the **derivativ**
The derivative $f'(x)$ a
- the gradient
- the gradient

To determine the gra
- the equation
- the x-coordi

The process of findin
about differentiation
gradient of a curve a

derivative: the derivative
differentiate: to find the

Rules for differentiati

The constant rule
The power rule
Sum rule
Difference rule

Note that all of the fo
$f'(x) = y' = \frac{dy}{dx} = \frac{d}{dx}[f(x)$

How would we calculate the gradient of the curve at a point, say at C? We cannot use the same calculation as for the average gradient, because we can't subtract the y and x-values of a single point. What we really want is the value of the gradient of the **tangent** to the curve at C.

New word

tangent: straight line that touches a curve at a single point

To find the gradient of the tangent we use a special calculation that finds the value of the change in the y-values as the change in the x-values approaches zero. This is called the **derivative** of the function and is indicated by $f'(x)$ or $\frac{dy}{dx}$.

The derivative $f'(x)$ at any point x on the curve of $f(x)$ gives:

- the gradient of the curve at that point
- the gradient of the tangent to the curve at that point.

To determine the gradient of a curve at a particular point, we need:

- the equation of the curve
- the x-coordinate of the point.

The process of finding the derivative is called **differentiation**. You will learn more about differentiation in Topic 7. For now, you will use differentiation to find the gradient of a curve at a point.

New words

derivative: the derivative of $f(x)$ at the point x is equal to the gradient of the tangent to $f(x)$ at x

differentiate: to find the derivative of a function

Rules for differentiation

The constant rule	The derivative of any constant number is zero.	Example: $f'(6) = 0$
The power rule	If $f(x) = kx^n$, then $f'(x) = n \times kx^{n-1}$	This works for any power: positive, negative or a fraction.
Sum rule	If $h(x) = f(x) + g(x)$, then $h'(x) = f'(x) + g'(x)$	Differentiate each term separately.
Difference rule	If $h(x) = f(x) - g(x)$, then $h'(x) = f'(x) - g'(x)$	Differentiate each term separately.

Note that all of the following can be used to indicate differentiation:

$$f'(x) = y' = \frac{dy}{dx} = \frac{d}{dx}[f(x)] = Df(x) = D\,y$$

Worked example 1

1 Find the derivatives of the following functions.

a) $f(x) = -x^3 + 3x^2 + 9$

b) $g(x) = 2(x^4 - x) + 9$

2 Determine:

a) $\frac{d}{dx}[x(2x - 1)(3x + 5)]$

b) $h'(x)$ if $h(x) = x^3 - 6x^2 + 9x + 16$

Answers

1 a) $f'(x) = -3x^2 + 6x$

b) $g(x) = 2(x^4 - x) + 9$

$g(x) = 2x^4 - 2x + 9$ Write the equation in standard form.

$\therefore g'(x) = 8x^3 - 2$

2 a) $\frac{d}{dx}[x(2x - 1)(3x + 5)]$

$= \frac{d}{dx}[x(6x^2 + 7x - 5)]$

$= \frac{d}{dx}(6x^3 + 7x^2 - 5x)$ Write the equation in standard form.

$= 18x^2 + 14x - 5$

b) $h'(x)$ if $h(x) = x^3 - 6x^2 + 9x + 16$

$h'(x) = 3x^2 - 12x + 9$

Activity 1

1 Determine $\frac{dy}{dx}$.

a) $y = -x + 3$
b) $y = x^2 + \frac{1}{2}x$
c) $y = 4x^4$
d) $y = 3x^4 - 10$
e) $y = 4x^2 - 3x - 7$
f) $y = 3x^4 + 3x^3 + 20x$
g) $y = (x + 1)(x + 2)(x + 3)$
h) $y = 5x^6 - x^{\frac{1}{2}}$

The gradient at a point

The derivative of a function gives us an expression that we can use to find the gradient at a particular point. To calculate the gradient at a point, substitute the x-value of the point into the derivative.

Worked example 2

1 Use differentiation to find the gradient of $y = 5x + 6$ at the point where $x = 0$.

2 Determine the gradient of the tangent to the curve $f(x) = 5x^2 - 4x$ at the points where:

a) $x = 2$ b) $x = 5$.

Worked example 2 (continued)

3 Determine the gradient of the function $g(x) = x^3 + 4x^2 - 3x - 5$ at the points:
a) $x = -2$ b) $x = 3$

4 Given that $h(x) = x^2 - 5x - 4$, determine the value of x at the point on the graph where the gradient is equal to 0.

Answers

1 $\frac{d}{dx}(5x + 6) = 5$

This is a constant, so the gradient is 5 at every point on the function. (Without using differentiation, we can also see that this is a linear function with a gradient of 5).

2 $f(x) = 5x^2 - 4x$

$f'(x) = 10x - 4$

Find the derivative of $f(x)$.

a) Where $x = 2$, gradient $= f'(2)$

$f'(2) = 10(2) - 4 = 16$

Substitute $x = 2$ into the derivative.

So the gradient of $f(x)$ at $x = 2$ is 16.

b) Where $x = 5$, gradient $= f'(5)$

$f'(5) = 10(5) - 4 = 46$

So the gradient of $f(x)$ at $x = 5$ is 46.

3 $g(x) = x^3 + 4x^2 - 3x - 5$

$g'(x) = 3x^2 + 8x - 3$

a) Where $x = -2$, gradient $= g'(-2)$

$g'(-2) = 3(-2)^2 + 8(-2) - 3 = 12 - 16 - 3 = -7$

So the gradient of $g(x)$ at $x = -2$ is -7.

b) Where $x = 3$, gradient $= g'(3)$

$g'(3) = 3(3)^2 + 8(3) - 3 = 27 + 24 - 3 = 48$

So the gradient of $g(x)$ at $x = 3$ is 48.

4 $h(x) = x^2 - 5x - 4$

$h'(x) = 2x - 5$

gradient $= h'(x) = 2x - 5 = 0$

$x = 2\frac{1}{2}$

So the gradient of $h(x)$ is 0 at the point where $x = 2\frac{1}{2}$.

Note

- The gradient of a linear function is described by a constant.
- The gradient of a quadratic function is described by a linear expression.
- The gradient of a cubic function is described by a quadratic expression.

Note

- Where the gradient of the tangent to the curve is positive, the curve is increasing.
- Where the gradient of the tangent to the curve is negative, the curve is decreasing.

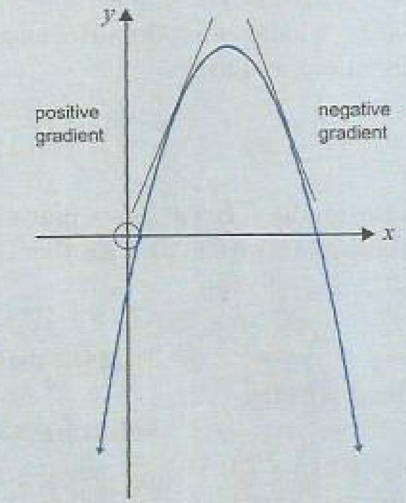

Figure 1.4

- A line with a gradient of 0 is a horizontal line. So in Worked example 2 Question 4, the horizontal tangent touches the graph where $x = 2\frac{1}{2}$ and this tells us that the point with the x-coordinate $2\frac{1}{2}$ is the turning point of the graph.

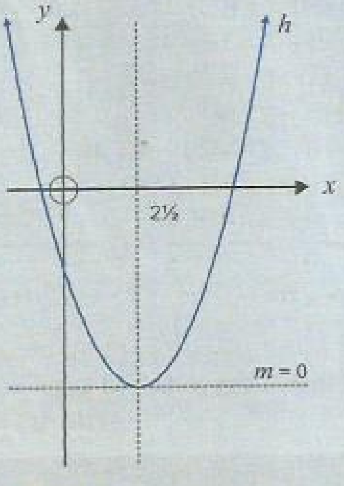

Figure 1.5

This point is also called the stationary point of the quadratic function: at this point, the gradient is equal to zero and the graph is neither decreasing nor increasing.

Activity 2

1 a) Determine th
where $x = 2$.
b) Is the graph
2 Calculate the gra
if $g(x) = -3x^2 + 5$
3 Given that $f(x) =$
graph where the
4 a) Calculate the
the point A w
b) Give the coo
5 Given that $g(x) =$
each of the follo
a) $x = \frac{1}{2}$
d) $x = 3$
6 Given $h(x) = x(x$
of the following
a) $x = -3$
d) $x = 0$

What is a cub

We know that a quad
A cubic function has t
For $f(x)$ to be a cubic f

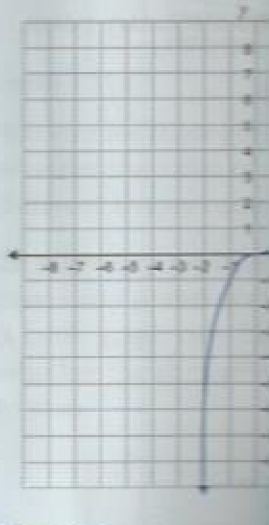

Figure 1.6

New wor

cubic function: an equa
power of 3.

Activity 2

1 a) Determine the gradient of the curve $f(x) = 4x^3 - 6x - 4$ at the point where $x = 2$.

b) Is the graph increasing or decreasing at this point?

2 Calculate the gradient of the tangent to the curve g at the point where $x = -4$ if $g(x) = -3x^2 + 5x + 2$.

3 Given that $f(x) = x^2 - 5x - 4$, determine the value of x at the point on the graph where the gradient is equal to -1.

4 a) Calculate the gradient of the tangent to the curve $y = (2x - 1)(x + 4)$ at the point A where $x = 1$.

b) Give the coordinates of A.

5 Given that $g(x) = x^3 - 6x^2 + 9x$, is g increasing, decreasing or stationary at each of the following x-values?

a) $x = \frac{1}{2}$ b) $x = 1$ c) $x = 2$
d) $x = 3$ e) $x = -1$ f) $x = -2$

6 Given $h(x) = x(x - 3)(2x + 4)$, is h increasing, decreasing or stationary at each of the following x-values?

a) $x = -3$ b) $x = -2$ c) $x = -1$
d) $x = 0$ e) $x = 1$ f) $x = 3$

What is a cubic function?

We know that a quadratic function has the standard form $f(x) = ax^2 + bx + c$.
A cubic function has the standard form $f(x) = ax^3 + bx^2 + cx + d$.
For $f(x)$ to be a cubic function, the formula must have the term x^3, so $a \neq 0$.

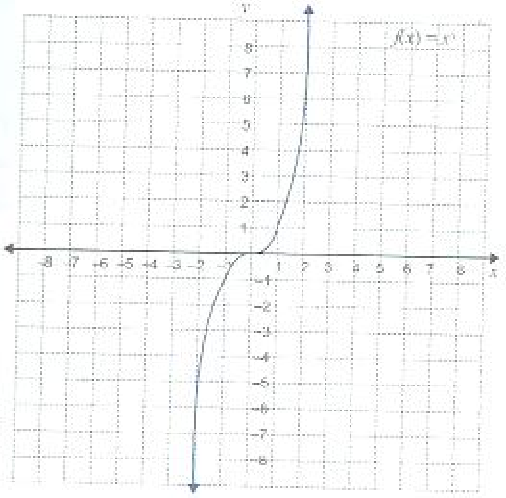

Figure 1.6

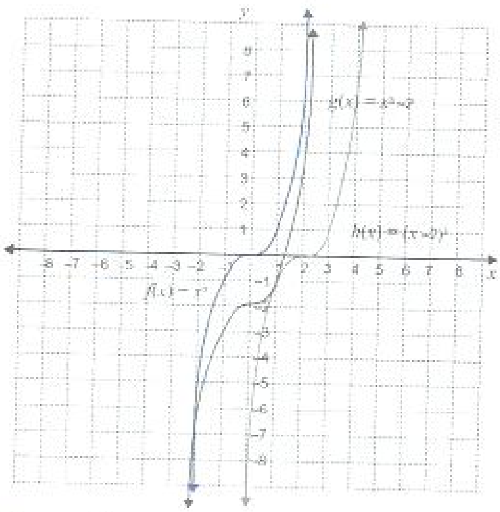

Figure 1.7

New word

cubic function: an equation having a power of 3.

Fig. 1.7 shows the graph of the basic function $f(x) = x^3$ with $g(x) = x^3 - 2$ being shifted two units down and $h(x) = (x - 2)^3$ being shifted two units to the right.

Note

The graph of $f(x) = x^3$ intersects the axes in only one place. Compare this to the graph of $f(x) = x^2$. What do you notice?

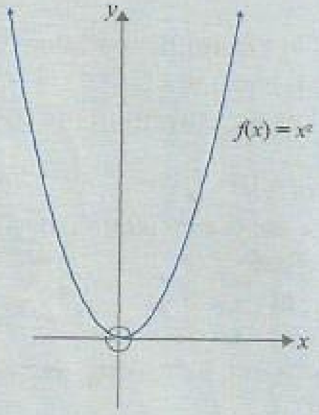

Figure 1.8

The features of cubic functions

The graph of a cubic function has:

- a y-intercept
- between one and three x-intercepts
- at most two stationary points, which may be the local minimum and the local maximum, or else the point of inflection
- one point of inflection, which may or may not also be a stationary point.

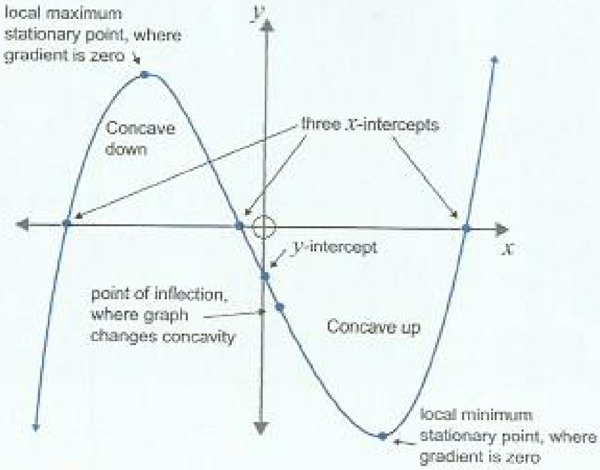

Figure 1.9

The value of *a* an

The first part of the gr

- If *a* is positive,

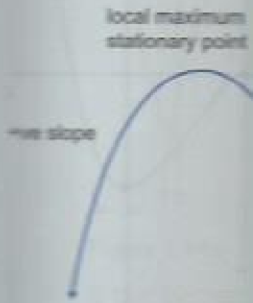

Figure 1.10a

- If *a* is negative,

Figure 1.11a

You can see there are fo
whether the value of *a*
different shapes come a

Stationary points

Stationary points are th
A cubic graph may hav
whether the derivative
to zero.

$) = x^3 - 2$ being
its to the right.

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tationary point.

The value of *a* and the shape of the graph

The first part of the graph of a cubic function $f(x)$ behaves as follows:

- If a is positive, as x increases the first part of the graph goes up.

$a > 0$

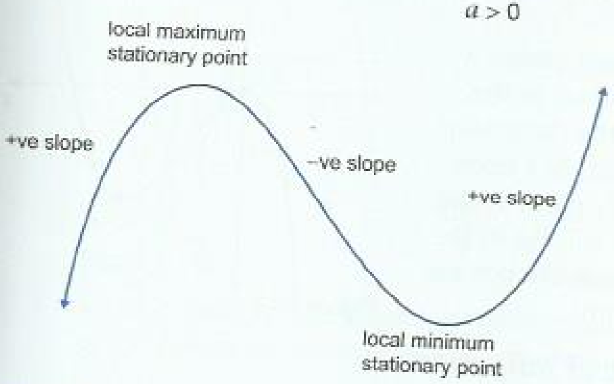


Figure 1.10a

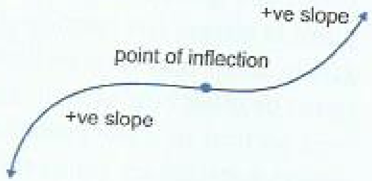


Figure 1.10b

- If a is negative, as x increases the first part of the graph goes down.

$a < 0$

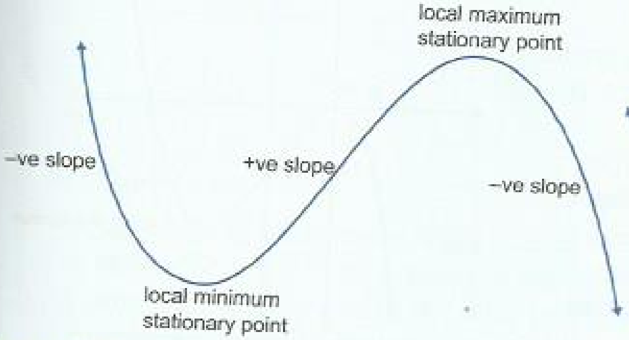


Figure 1.11a

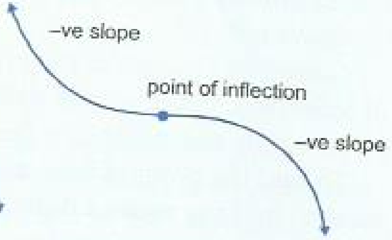


Figure 1.11b

You can see there are four possible shapes. The shape depends first of all on whether the value of a is positive or negative. Let's explore further how these different shapes come about.

Stationary points

Stationary points are the places on the graph where the gradient is equal to zero. A cubic graph may have a maximum of two stationary points. This depends on whether the derivative of the function has one or two solutions when it is equated to zero.

For example, Fig. 1.12 has stationary points at (0, 0) and (4, –4).

- The local maximum is at 0.
- The local minimum is at –4.

Note that a stationary point is not always a local minimum or maximum point. In the graph of a quadratic function, the stationary point is always the turning point. In a cubic graph, there may be a stationary point at the point of inflection as well. You will see in the next section that the point of inflection is not always a stationary point though!

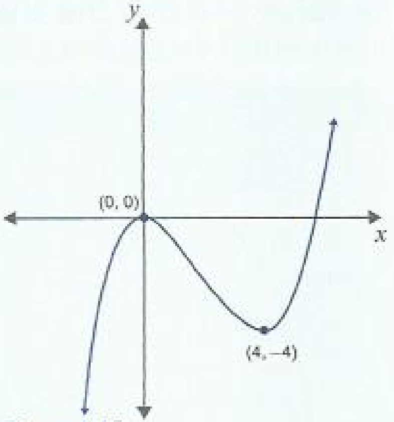


Figure 1.12

Concavity and the point of inflection

We have seen that the derivative tells us about the gradient of the curve. We can also describe the concavity of the graph. When the graph curves downwards, we say the graph is "concave down" and when the graph curves upwards, we say the graph is "concave up".

Concavity changes at the point of inflection. Cubic graphs always have exactly one point of inflection.

To find the point of inflection, we need to find the **second derivative** of the graph. This means the derivative of the derivative. The second derivative is indicated by $f''(x)$.

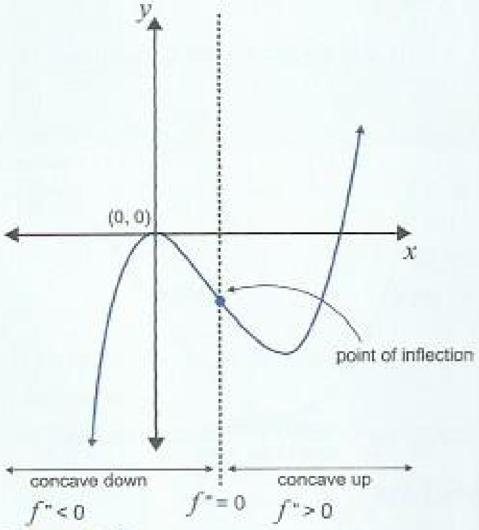


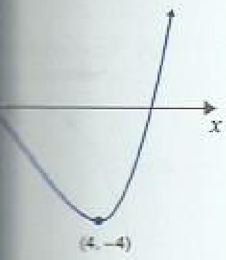
Figure 1.13

New word

second derivative: the derivative of a derivative

Shape of the graph

- The derivative of a cubic function is a quadratic function. The stationary points of the cubic graph occur at the x-values where the derivative has its roots. Figures 1.14 to 1.15 show the graph of a cubic function and its derivative, a quadratic function. Look carefully at the shapes of these graphs.



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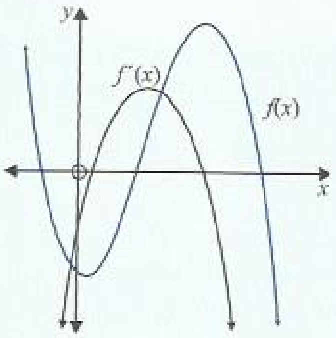


Figure 1.14a

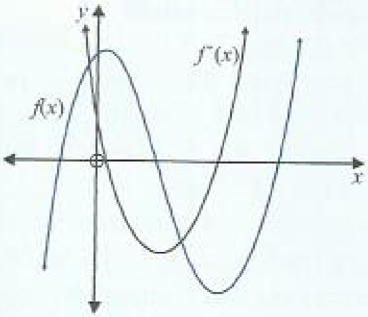


Figure 1.14b

- If the derivative (the quadratic function) has no real roots, then the cubic graph has no local minimum nor maximum.

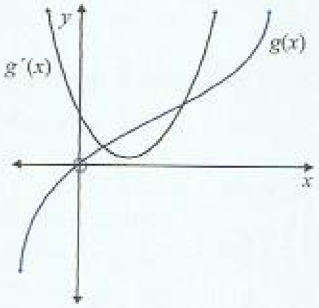


Figure 1.15a

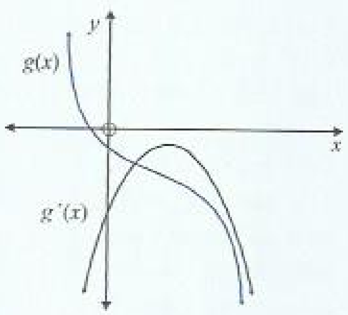


Figure 1.15b

- The point of inflection is at the x-value of the turning point of the derivative. If the turning point of the derivative touches the x-axis, then the point of inflection of the cubic graph is also the only stationary point. So for a point of inflection to be a stationary point, it means that the derivative has only one root.

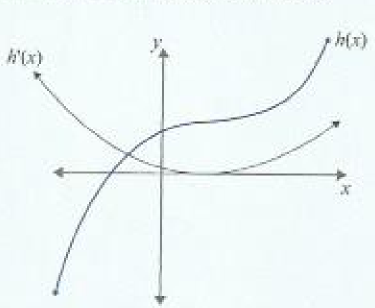


Figure 1.16a

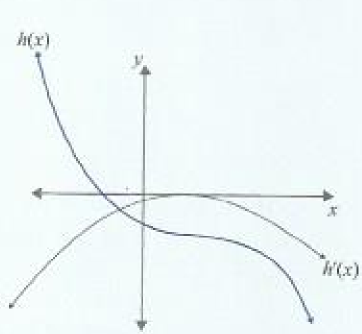


Figure 1.16b

In Figs. 1.15a and b there are no stationary points.
In Figs. 1.16a and b there is one stationary point.

Worked example 3

1 a) Is the graph of the quadratic function $g(x) = -x^2 + 5x - 4$ concave up or down? Test this by finding the second derivative of this quadratic function.

b) Draw a sketch graph of $g(x)$.

c) If $g(x)$ is the derivative of $h(x)$, sketch a possible graph of $h(x)$ on the same set of axes.

2 Find the second derivative of the cubic function $f(x) = x^3 - 3x^2 - 24x + 8$. Using the a-value and the second derivative, indicate where $f(x)$ will be concave up and concave down and what the x-coordinate of the point of inflection is.

Answers

1 a) a is negative, so the graph $g(x) = -x^2 + 5x - 4$ turns downwards, i.e. it is concave down.

$g'(x) = -2x + 5$

$g''(x) = -2$

The second derivative is negative, therefore we can confirm it is concave down.

b) Sketch graph of $g(x) = -x^2 + 5x - 4$:

The turning point occurs where $g'(x) = -2x + 5 = 0$

so $x = 2\frac{1}{2}$ and $y = -(2\frac{1}{2})^2 + 5(2\frac{1}{2}) - 4 = 2\frac{1}{4}$

Find the zeros:

$0 = x^2 - 5x + 4$

$0 = (x - 4)(x - 1)$

Zeros are $x = 4$ and $x = 1$

c) We don't know the formula for $h(x)$, so we can only infer the general shape in relation to its derivative, $g(x)$.

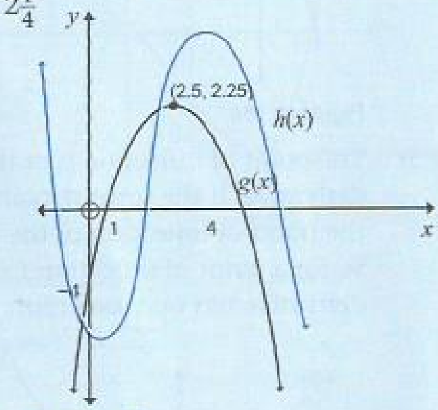


Figure 1.17

2 $f(x) = x^3 - 3x^2 - 24x + 8$

a is positive, so we know we are working with a graph that looks like this:

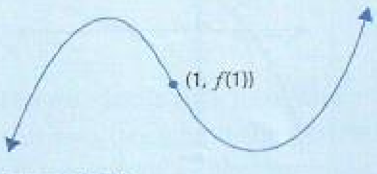


Figure 1.18

$f'(x) = 3x^2 - 6x - 24$

$f''(x) = 6x - 6$

So the point of inflection of $f(x)$ occurs where $f''(x) = 6x - 6 = 0$

$\therefore$ at $x = 1$

So $f(x)$ is concave down where $x < 1$ and concave up where $x > 1$.

Activity 3

1 For each of these quadratic functions:
  (i) Use differentiation to show whether they are concave up or down.
  (ii) Determine the coordinates of the turning point.
    a) $f(x) = (x - 3)^2 - 8$
    b) $g(x) = -x^2 - 2x - 6$
    c) $h(x) = x^2 + 4x - 5$

2 For each function:
  (i) Determine the stationary (turning) point(s).
  (ii) Calculate the coordinates of the point of inflection.
  (iii) Draw a rough sketch of the shape of the graph, showing where it is concave up and where it is concave down.
    a) $f(x) = x^3 + x^2 - 12$
    b) $g(x) = -x^3 - 3x^2 + 9x - 5$

The zeros of a cubic function

So far we have worked out how to determine the shape of a cubic graph. We still need to determine the x-intercepts of the graph.

What is a real zero?

There are three possible situations when you find the zeros of a quadratic function $f(x) = ax^2 + bx + c$, depending on the value of the discriminant (Δ or $b^2 - 4ac$).

A quadratic function always has either no real zeros or two real zeros (which may be equal). This gives us three possibilities, shown below:

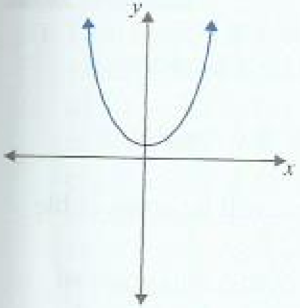


Figure 1.19a
No real zeros
$\Delta < 0$

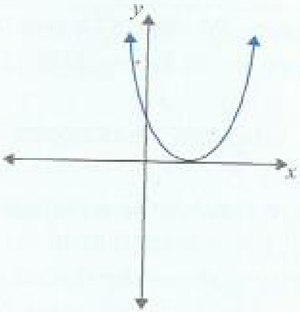


Figure 1.19b
Two equal real zeros
$\Delta = 0$

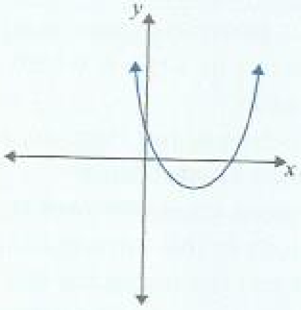


Figure 1.19c
Two unequal real zeros
$\Delta > 0$

Note

To solve an equation in the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, we take the square root of $b^2 - 4ac$. The square root of a negative number is not a real number, so the equation has no real solutions if $b^2 - 4ac < 0$. This means that the parabola does not intersect with the x-axis.

A cubic function has either one real zero, or three real zeros. Unlike a quadratic equation, which may have no real solution, a cubic function always has at least one real zero. If a cubic function does have three zeros, two of them may be equal. This gives us the possibilities shown below:

New words

zeros: the values of x where $f(x) = 0$

solutions: the points where $f(x) = 0$

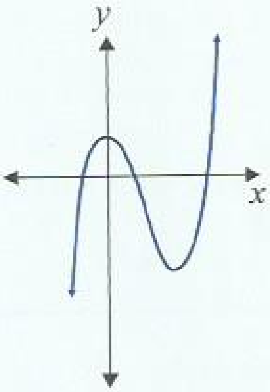


Figure 1.20a: Three real roots, all different

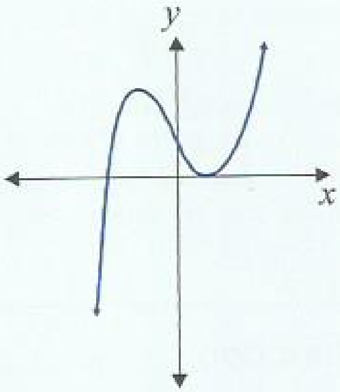


Figure 1.20b: Three real roots (two equal)

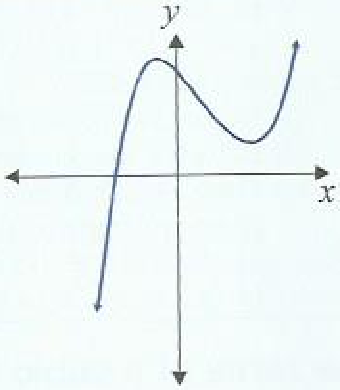


Figure 1.20c: One real root

Finding the zeros of a cubic equation

First write the function in its standard form $f(x) = ax^3 + bx^2 + cx + d$. Then set the cubic function equal to zero, in other words $ax^3 + bx^2 + cx + d = 0$. Remember that because there is always at least one real root, the cubic equation $ax^3 + bx^2 + cx + d = 0$ will have at least one factor.

This means that a cubic equation can always be written in the form $(x + p)(x^2 + qx + r) = 0$, which is one linear factor multiplied by a quadratic expression.

Whether or not there are any other real roots depends on the remaining quadratic expression.

Because you know how to solve a quadratic equation, you will be given cubic functions in this form or in the fully factorised form.

To find the rest of the information about the stationary points and point of inflection, you need to write the equation in the form $f(x) = ax^3 + bx^2 + cx + d$, by multiplying out the factors.

Worked example 4

1 Give the zeros of each function.

a) $f(x) = x(x + 3)^2$

b) $f(x) = (x - 1)(x - 2)(x - 3)$

c) $f(x) = (x - 1)(2x^2 - x - 3)$

d) $f(x) = (x + 5)(x^2 + 9x + 20)$

e) $f(x) = (x - 1)(x^2 + 2x + 3)$

f) $f(x) = x(x^2 - 81)$

Worked example 4 (continued)

Answers

1 a) Set $f(x)$ equal to zero: $x(x+3)^2 = 0$
$x(x+3)(x+3) = 0$
Solve: $x = 0$ or $x = -3$ (repeated)

b) $(x-1)(x-2)(x-3) = 0$
$\therefore x = 1, x = 2$ or $x = 3$

c) $(x-1)(2x^2 - x - 3) = 0$
Factorise the quadratic expression $2x^2 - x - 3$:
$(x-1)(2x-3)(x+1) = 0$
$\therefore x = 1, x = \frac{3}{2}$ or $x = -1$

d) $(x+5)(x^2 + 9x + 20) = 0$
$(x+5)(x+4)(x+5) = 0$
$\therefore x = -5$ (repeated) or $x = -4$

e) $(x-1)(x^2 + 2x + 3) = 0$
The quadratic expression $x^2 + 2x + 3$ cannot be factorised.
Check the discriminant: $\Delta = b^2 - 4ac = 2^2 - 4(1)(3) = -8$
Because $\Delta < 0$, the quadratic equation has no real roots.
The cubic equation therefore has only one real root, $x = 1$.

f) $x(x^2 - 81) = 0$
$x(x-9)(x+9) = 0$ (difference of squares)
$\therefore x = 0$ or $x = 9$ or $x = -9$

Activity 4

1 Find the zeros of the following cubic equations. State whether a root is repeated and how many times.

a) $(x-5)(x^2 - 10x + 25) = 0$
b) $(x-3)^2(x+4) = 0$
c) $(x+2)(x^2 + 7x + 10) = 0$
d) $(x+1)(x^2 - 12x + 20) = 0$
e) $(x+2)(x^2 + 6x + 10) = 0$
f) $(x+1)(x-2)(x-3) = 0$

2 Write down the coordinates of the x- and y-intercepts for these functions:

a) $f(x) = (x+2)^2(x-4)$
b) $f(x) = x(x-7)^2$
c) $f(x) = (x-4)^3$
d) $f(x) = x^3 - 3x^2 + 3x$
e) $f(x) = x^3 - 12x^2$
f) $f(x) = x(x^2 - 64)$
g) $f(x) = x^3 + 6x^2$
h) $f(x) = x^3 - 6x^2 + 9x$
i) $f(x) = -x^3 + 3x^2 - 2x$
j) $f(x) = (x-2)(3x^2 - 6x + 7)$

Draw graphs of cubic functions

We need to consider more features when drawing graphs of cubic functions than for quadratic functions. There could be two stationary points, and we always need to know where the point of inflection is. We also need the y- and x-intercepts. We learnt how to find all of these in the previous section.

How to draw a sketch graph of a cubic function

Step 1: Consider the sign of a to determine the general shape of the graph.

Step 2: Give the y-intercept (this is the value of d when the equation is in standard form $f(x) = ax^3 + bx^2 + cx + d$).

Step 3: Calculate the x-intercept(s) by setting $f(x) = 0$ and solving the equation.

Step 4: Calculate the coordinates of the stationary points (turning points) by calculating $f'(x)$ and setting it equal to 0 to find x. Substitute back into **the original** equation to find the y-value.

Step 5: Calculate the coordinates of the point of inflection by calculating $f''(x)$ and setting it equal to 0 to find x. Substitute back into **the original** equation to find the y-value.

Step 6: Plot a few sample points (you may need these if some of the points you calculate are the same points).

Tip: Drawing a sketch graph means that you need to show the general shape of the graph correctly and indicate all the features. It is not an accurate plot of the graph.

Worked example 5

1 Sketch the graph of $f(x) = x(x + 2)^2$.

2 Sketch the graph of $f(x) = (x - 1)(x - 2)(x - 3)$.

Answers

1 Shape:

a is positive and there are two x-intercepts, so the shape is:

or

y-intercept:

$f(x) = x(x + 2)^2 = x^3 + 4x^2 + 4x + 0$ Write the equation in standard form.

$d = 0$ ∴ y-intercept is at $(0, 0)$

x-intercept(s):

$x(x + 2)^2 = 0$ Set $f(x) = 0$

$x(x + 2)(x + 2) = 0$

∴ the x-intercepts are at $(0, 0)$, $(-2, 0)$

Worked example 5 (continued)

Stationary point(s):

$f(x) = x^3 + 4x^2 + 4x$

$f'(x) = 3x^2 + 8x + 4 = 0$ Set $f'(x)= 0$

$(3x + 2)(x + 2) = 0$

$\therefore x = -\frac{2}{3}$ or $x = -2$

Substitute back into $f(x)$.

Substitute $x = -\frac{2}{3}$

$y = x(x + 2)^2$

$y = -\frac{2}{3}(1\frac{1}{3})^2 = -\frac{32}{27} = -1\frac{5}{27}$

Stationary point is $(-\frac{2}{3}, -1\frac{5}{27})$

Substitute $x = -2$

$y = x(x + 2)^2$

$y = -2(0)^2 = 0$

Stationary point is (–2, 0), which is the same as the x-intercept.

Point of inflection:

$f'(x) = 3x^2 + 8x + 4$

$f''(x) = 6x + 8$

$0 = 6x + 8$ Set $f''(x) = 0$

$x = -1\frac{1}{3}$

Substitute $x = -1\frac{1}{3}$ back into the original equation

$y = -1\frac{1}{3}(-1\frac{1}{3} + 2)^2$

$y = -\frac{4}{3}(\frac{2}{3})^2 = -\frac{16}{27}$

Point of inflection is $(-1\frac{1}{3}, -\frac{16}{27})$

Draw the sketch graph neatly, labelling the points.

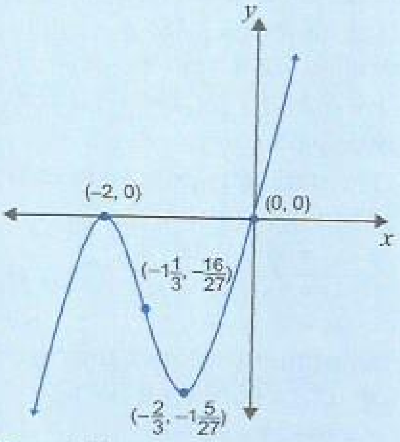


Figure 1.21

2 Shape:

a is positive and there are three x-intercepts, so the shape is:

y-intercept:

$f(x) = (x - 1)(x - 2)(x - 3)$

$= (x - 1)(x^2 - 5x + 6)$

$= x^3 - 6x^2 + 11x - 6$

$\therefore$ the y-intercept is at (0, –6)

x-intercept(s):

$(x - 1)(x - 2)(x - 3) = 0$

$x = 1, x = 2$ or $x = 3$

$\therefore$ the x-intercepts are at (1, 0),(2, 0), (3, 0)

Worked example 5 (continued)

Stationary point(s):

$f(x) = x^3 - 6x^2 + 11x - 6$

$f'(x) = 3x^2 - 12x + 11$

$f'(x) = 3x^2 - 12x + 11 = 0$ Set $f'(x) = 0$.

Use the quadratic formula to solve:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{12 \pm \sqrt{12^2 - 4(3)(11)}}{2(3)}$$

$$= \frac{12 \pm \sqrt{144 - 132}}{6}$$

$$= \frac{12 \pm \sqrt{12}}{6}$$

$\therefore x = 2.58$ or $x = 1.42$

Substitute $x = 2.58$

$y = x^3 - 6x^2 + 11x - 6$

$y = (2.58)^3 - 6(2.58)^2 + 11(2.58) - 6$

$y = -0.38$

Substitute $x = 1.42$

$y = x^3 - 6x^2 + 11x - 6$

$y = (1.42)^3 - 6(1.42)^2 + 11(1.42) - 6$

$y = 0.38$

The stationary points are (2.58, –0.38) and (1.42, 0.38).

Point of inflection:

$f'(x) = 3x^2 - 12x + 11$

$f''(x) = 6x - 12 = 0$ Set $f''(x) = 0$.

$x = 2$

Substitute $x = 2$ back into the original equation

$y = (2)^3 - 6(2)^2 + 11(2) - 6$

$y = 8 - 24 + 22 - 6$

$y = 0$

The point of inflection is at (2, 0), which is one of the x-intercepts.

Draw the sketch graph neatly, labelling the points.

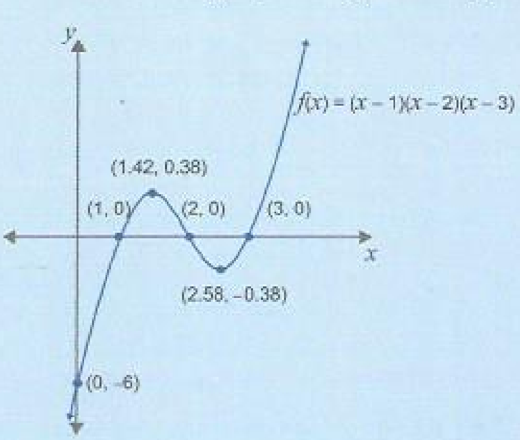


Figure 1.22

Support activity

Copy the graph of the
books and complete th

Step 1: The a-value determines the shape of the graph (sketch):

Step 2: What is the y-interce
Multiply out:
$f(x) = -2x^3 + 5x^2 + 4x\boxed{-3}$
Mark the point on the graph.

Step 3: What are the x-inter
Factorise the quadratic part:
$0 = -(x + 1)(2x^2 - 7x + 3)$
$0 = -(x + 1)(2x$ ____ $)(x$ ____ $)$
Mark the x-intercepts on the graph.

Figure 1.23

Activity 5

1 Draw a sketch grap
- The graph inte
- $f'(-1) = f'(1) =$
- $f(1) = -4$; $f(0) =$
- $f'(x) > 0$ if $x <$
- $f'(x) < 0$ if $-1 <$

2 The graph in Fig.
A, C and E are inte
B and D are turnin
a) Calculate the
b) Calculate the
c) Write down th
d) For which valu
be decreasing?
e) Find the coord
of inflection.
f) Is this a station
Explain.

Support activity

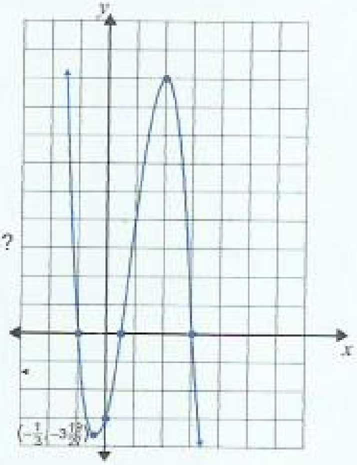
Copy the graph of the function $f(x) = -(x + 1)(2x^2 - 7x + 3)$ into your exercise books and complete the labels.

Step 1: The a-value determines the shape of the graph (sketch):

Step 2: What is the y-intercept?
Multiply out:
$f(x) = -2x^3 + 5x^2 + 4x \boxed{-3}$
Mark the point on the graph.

Step 3: What are the x-intercepts?
Factorise the quadratic part:
$0 = -(x + 1)(2x^2 - 7x + 3)$
$0 = -(x + 1)(2x \_\_\_)(x\_\_\_)$
Mark the x-intercepts on the graph.

$f(x) = -(x + 1)(2x^2 - 7x + 3)$



Step 4: Differentiate
$f(x) = 2x^3 + 5x^2 + 4x - 3$
$f'(x) = -6x^2 + \_\_\_ + \_\_\_$
Set $f'(x) = 0$ and solve.
Find the y-values by substituting into f.
Mark the two stationary points on the graph.

Step 5: Differentiate again.
$f''(x) = \_\_\_x + \_\_$
Set $f''(x) = 0$.
Determine the point of inflection.
Do you get (0.83, 2.65?)
If not, try again.
Mark the point on the graph.

Join all the points!

Figure 1.23

Activity 5

1 Draw a sketch graph of $f(x) = ax^3 + bx^2 + cx + d$, given the following information:
- The graph intersects the x-axis at $x = -1$ and $x = 2$.
- $f'(-1) = f'(1) = 0$
- $f(1) = -4$; $f(0) = -2$
- $f'(x) > 0$ if $x < -1$ or $x > 1$
- $f'(x) < 0$ if $-1 < x < 1$

2 The graph in Fig. 1.24 represents $g(x) = x^3 - 3x^2 - 4x$.
A, C and E are intercepts with the axes.
B and D are turning points of the graph.
- a) Calculate the coordinates of D.
- b) Calculate the coordinates of A, C and E.
- c) Write down the length of AE.
- d) For which values of x will the graph be decreasing?
- e) Find the coordinates of the point of inflection.
- f) Is this a stationary point? Explain.

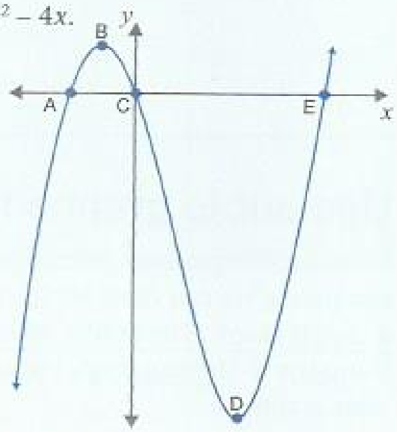


Figure 1.24

Activity 5 (continued)

3 Sketch the following graphs.

a) $f(x) = (x - 5)(x^2 - 10x + 25)$
b) $f(x) = (x - 3)^2(x + 4)$
c) $f(x) = (x + 2)(x^2 + 7x + 10)$
d) $f(x) = -(x + 1)(x^2 - 12x + 20)$
e) $f(x) = (x + 2)(x^2 + 6x + 10)$
f) $f(x) = -(x + 1)(x - 2)(x - 3)$

*Extension activity

1 The graph of $g'(x)$ is shown in Figure 1.25.

a) What is the formula for $g'(x)$?
b) Write down a general formula for the gradient of a tangent to $g'(x)$.
c) For which values of x is $g(x)$ increasing?
d) For which values of x is $g(x)$ decreasing?
e) Give the x-values of the stationary points of $g(x)$.
f) If $g(x)$ intersects the y-axis at 10, determine the formula for $g(x)$.
g) Find the point of inflection of $g(x)$.

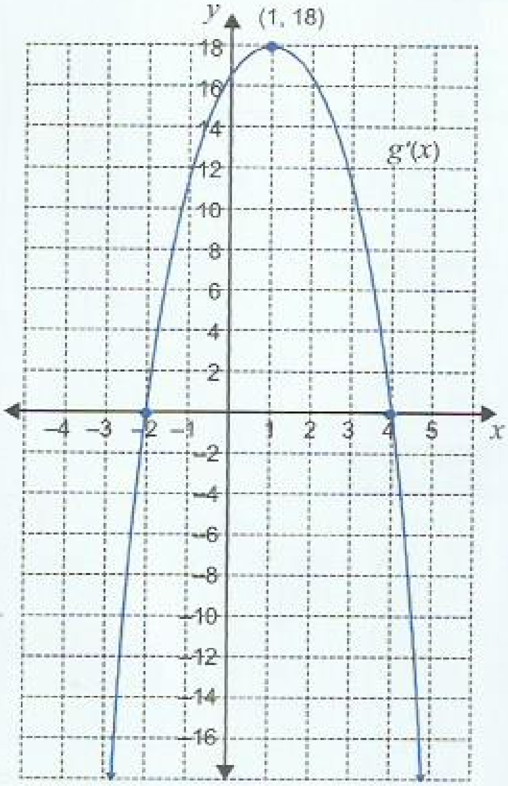


Figure 1.25

Use cubic graphs to find solutions

Some cubic equations cannot be factorised. To solve these kinds of cubic equations, we can draw an accurate graph of the cubic expression rather than a sketch graph. The points where the graph crosses the x-axis will give you the solutions to the equation, but their accuracy will be limited to the accuracy of your graph.

$(x + 4)$

$1)(x^2 - 12x + 20)$

$1)(x - 2)(x - 3)$

kinds of cubic
pression rather than
axis will give you the
ed to the accuracy of

Worked example 6

Solve $x^3 + 4x^2 + x - 5 = 0$.

Answer

We are not able to solve this equation by factorisation. So we draw a graph of $y = x^3 + 4x^2 + x - 5$ as accurately as possible and read off the solutions.

y-intercept: $y = -5$

Stationary points: $\frac{dy}{dx} = 3x^2 + 8x + 1 = 0$

This can't be factorised, so use the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = -0.13$ or $x = -2.54$

Substitute into the original equation: $y = -5.06$ $\quad y = 1.879$

Table of values:

x	–5	–4	–3	–2	–1	0	1	2	3	4	5
y	–35	–9	1	1	–3	–5	1	21	61	127	225

Calculate the point of inflection:

$f''(x) = \frac{dy}{dx}(3x^2 + 8x + 1) = 0$

$6x + 8 = 0$

$\therefore \quad x = -\frac{4}{3}$

Substitute back into $f(x)$:

$y = x^3 + 4x^2 + x - 5$

$y = (\frac{-4}{3})^3 + 4(\frac{-4}{3})^2 + (\frac{-4}{3}) - 5$

$y = -1\frac{16}{27}$

$\therefore$ the point of inflection is $(-\frac{4}{3}, -1\frac{16}{27})$

For the graph, focus on the area around the axes and the stationary points to increase the accuracy of reading off the roots.

The graph intersects the x-axis at three places, so there are three real roots.

The accuracy will be limited to the accuracy of the graph.

From the graph we find the approximate solutions $x \approx -3.2; -1.7; 0.9$

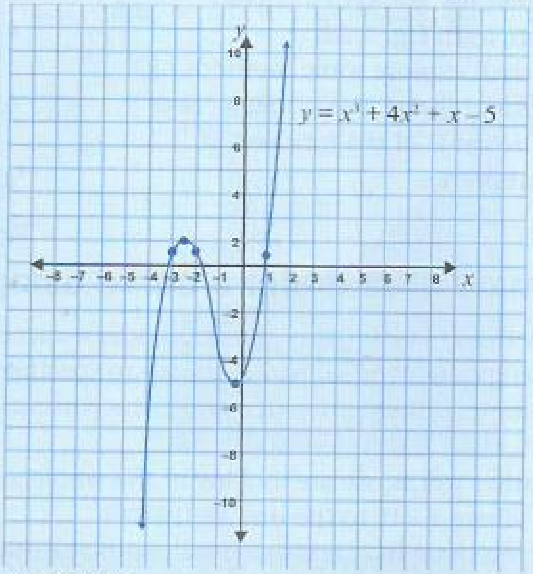


Figure 1.26

Activity 6

1 Use graphs to solve the following cubic equations as accurately as you can.

a) $x^3 - 4x^2 - 6x + 5 = 0$

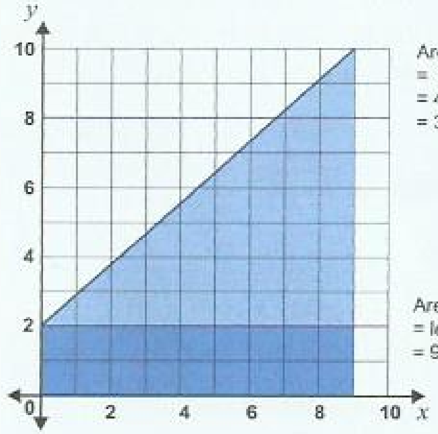
b) $2x^3 - 3x^2 - 4x - 35 = 0$

c) $x^3 - 3x^2 - x + 1 = 0$

d) $x^3 + 2x^2 + 3x - 5 = 0$

The area under a curve

To solve certain problems, we need to find the area between a graph and the horizontal axis. This is easy if the graph is a straight line.



Area of triangle
= ½ base x height
= 4.5 x 8
= 36 square units

Area of rectangle
= length x breadth
= 9 x 2 = 18 square units

Figure 1.27

The total area is 36 + 18 = 54 square units.

It is more complicated to calculate the area under the graph of a curved function, as in Fig. 1.28.

We shall explore some methods of approximating the area under a curve using the value of the function at different points.

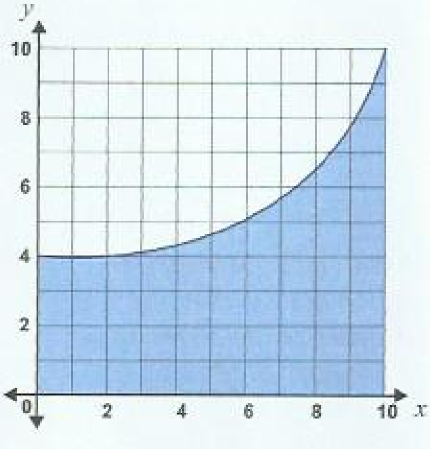


Figure 1.28

Estimate area under a curve using rectangles

One method of estimating the area under a curve is by using sums of the areas of rectangles with equal width, drawn to specified points on the function.

Example: Estimate the area under the curve in Fig. 1.28 in the interval from 0 to 10.

In other words, we are looking at the whole area under the curve, from the y-axis, where $x = 0$, to the line $x = 10$.

We can draw five rectangles each with a width of 2 units, i.e. $\Delta x = 2$ units.

Note

Δx (delta x) means the change or difference in the value of x. Don't confuse it with Δ, the discriminant ($b^2 - 4ac$) in a quadratic expression.

We can draw these rectangles in three ways: Fig. 1.29a focuses on the left endpoint of each rectangle, Fig. 1.29b focuses on the right endpoint of each and Fig. 1.29c on the midpoint of each rectangle.

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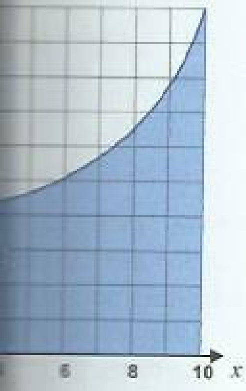

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sums of the areas
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confuse it with Δ, the

on the left endpoint
ach and Fig. 1.29c

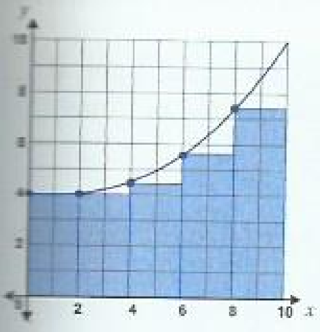

Figure 1.29a
Left endpoint

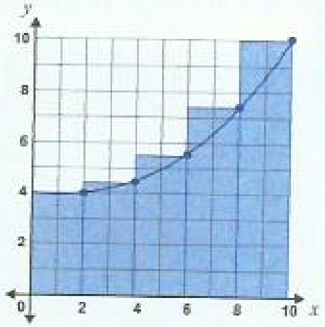

Figure 1.29b
Right endpoint

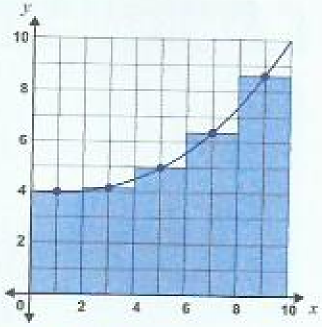

Figure 1.29c
Midpoint

The interval is $x \in [0, 10]$. There are 5 sub-intervals with $\Delta x = 2$.

- In Fig. 1.29a, we use the function value at the left endpoint of the sub-interval, Δx, as the height of the rectangle. This underestimates the area, as we can see from the white areas under the graph.
- In Fig. 1.29b, we use the function value at the right endpoint of the sub-interval, Δx, as the height of the rectangle. This tends to overestimate the area, as seen on the graph.
- In Fig, 1.29c, we use the function value at the midpoint of the sub-interval, Δx, as the height of the rectangle. This seems to be a better approximation, but it might not always be so.

You can see that if we divide the horizontal distance into smaller Δx-values, our approximation will be better. In fact, the closer the value of Δx gets to 0, the better the estimate will be.

In Fig. 1.30, we've drawn more rectangles, so $\Delta x = 1$ unit instead of 2 units, so there are now 10 rectangles in total.

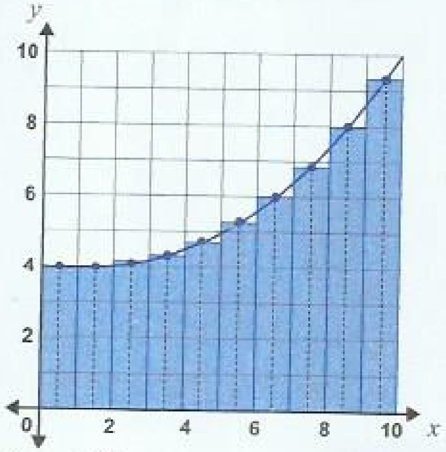

Figure 1.30

Think about it

How many rectangles would you need to make sure that the rectangles fit perfectly under the curve? You would need to approach an infinite number.

In Topic 7, you will learn about definite integrals, where the idea of finding the limit of the area as the number of rectangles approach infinity (and the width Δx approaches 0) is used to calculate area accurately.

Worked example 7

Estimate the area between $f(x) = x^3 - 5x^2 + 6x + 5$ and the x-axis in the interval 0 to 4, using $n = 5$ intervals. (This part of the curve is positive.) Use all three cases of the sum of areas of rectangles.

Answer

The width of each sub-interval will be $4 \div 5 = 0.8$. This means that the endpoints of the subintervals are: 0; 0.8; 1.6; 2.4; 3.2; 4.

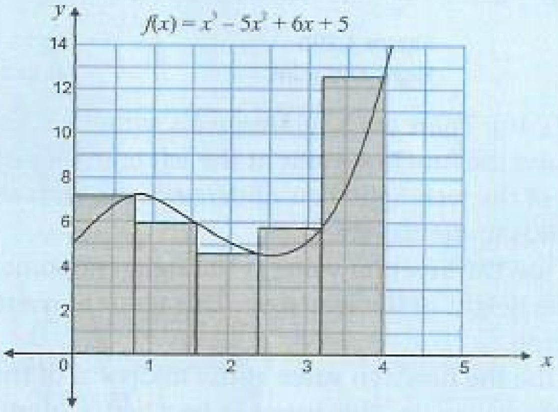

Figure 1.31

Draw up a table of values of the function at each endpoint.

x	0	0.8	1.6	2.4	3.2	4
$f(x)$	5	7.112	5.896	4.424	5.768	13

Then calculate the sum of the areas for each case.

Left endpoints:

Area = $0.8\,f(0) + 0.8\,f(0.8) + 0.8\,f(1.6) + 0.8\,f(2.4) + 0.8\,f(3.2) = 22.56$ square units

Right endpoints:

Area = $0.8\,f(0.8) + 0.8\,f(1.6) + 0.8\,f(2.4) + 0.8\,f(3.2) + 0.8\,f(4) = 28.96$ square units

Midpoints:

Area = $0.8\,f(0.4) + 0.8\,f(1.2) + 0.8\,f(2) + 0.8\,f(2.8) + 0.8\,f(3.6) = 25.12$ square units

Activity 7

Use the sum of areas of rectangles method to solve these problems.

1 a) Find the area under the curve $f(x) = 1 - x^2$ between $x = 0.5$ and $x = 2$, for $n = 5$.

b) Find the area under the curve given in 1a), but this time use $n = 10$.

2 An arch has the shape of a curve $f(x) = 2x^2 + 3$ between $x = 0$ and $x = 4$ and the x-axis, for $n = 5$. Find the area under the arch.

3 Find the area under the curve $f(x) = 2x^3 + 4$ between $x = 0$ and $x = 8$ and the x-axis, for $n = 5$.

-axis in the interval
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ns that the endpoints

2.4	3.2	4
4.424	5.768	13

) = 22.56 square units

) = 28.96 square units

) = 25.12 square units

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= 0.5 and $x = 2$,

time use $n = 10$.
= 0 and $x = 4$ and

0 and $x = 8$ and the

Estimate the area under a curve using trapeziums

We can use trapezium shapes to estimate the area under a curve. The trapeziums are drawn with their parallel sides perpendicular to the x-axis.

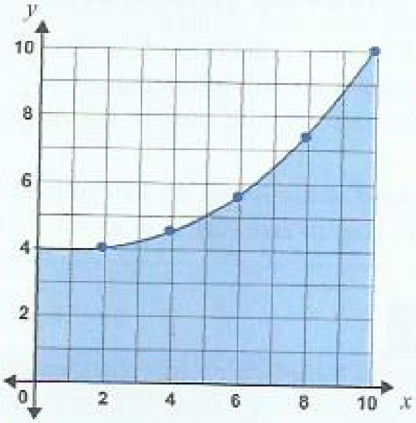

Figure 1.32

The area of the trapeziums averages the area of the left and right rectangles we used previously.

The trapezium rule is derived from the formula for the area of a trapezium:

$$\text{Area} = \frac{h}{2}[y_0 + y_n + 2(y_1 + y_2 + \ldots + y_{n-1})]$$

where h is the height of the trapeziums (the size of the sub-intervals chosen) and the y-values are the lengths of the parallel sides at the end of each strip. The y-values are equal to the function values.

Worked example 8

Find the area between the x-axis and the curve $f(x) = x^2 + 2$ between $x = 1$ and $x = 5$ using the trapezium rule with four trapeziums.

Answer

$\text{Area} = \frac{h}{2}[y_0 + y_n + 2(y_1 + y_2 + \ldots + y_{n-1})]$

To find the y-values, you could draw up a table of values.

x	1	2	3	4	5
$y = f(x)$	3	6	11	18	27

$h = 1$, so $\text{Area} = \frac{1}{2}(3 + 27 + 2(6 + 11 + 18)) = 50$ square units.

Activity 8

Solve the same problems as in Activity 7, but this time, use the trapezium rule.

SUB-TOPIC 2 Inverse functions

Draw graphs of inverse functions

The inverse of a function, $f^{-1}(x)$ is a reflection of $f(x)$ in the line $y = x$. To obtain the inverse, the x and y variables are interchanged.

The inverse of a function exists if, and only if, the original function is a one-to-one mapping.

If a function is a many-to-one mapping, we can restrict the domain of the function in such a way that the inverse function exists. This restriction is often of the form $x \leq 0$, or $x \geq 0$.

Worked example 9

Draw the graph of the function $f(x)$ which has the formula $y = x^2 + 2x - 3$.

Find the formula for the inverse of $f(x)$ and restrict the domain such that the inverse is a function. Draw the inverse function on the same set of axes.

Answer

$y = x^2 + 2x - 3$

Inverse: $x = y^2 + 2y - 3$

In Fig. 1.33a, the domain has not been restricted to find the inverse. The inverse is not a function, as there are two y-values for each x-value.

In Fig. 1.33b, the domain of the function has been restricted to $x \leq -1$ to find the inverse.

$y = x^2 + 2x - 3$

Inverse: $x = y^2 + 2y - 3$

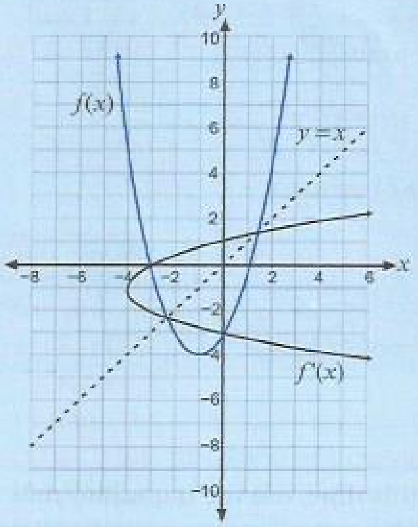

Figure 1.33a

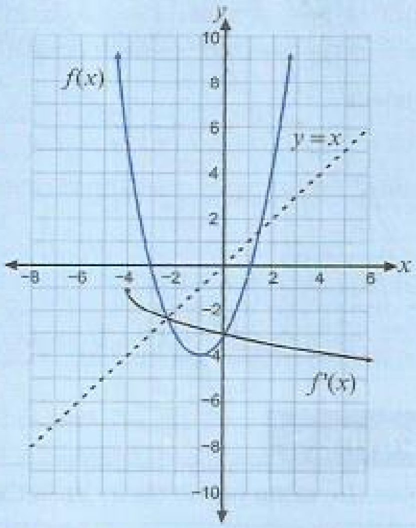

Figure 1.33b

Activity 9

For each of the follo

a) Give the equatio

b) If necessary, rest
function.

c) On the same set

1 $f(x) = -2x - 2$

2 $2x - y + 3 = 0$

3 $g(x) = \frac{1}{2}x^2$

4 $p(x) = -x^2$

Exponential gr

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Did you know?

Exponential graphs are
rapid growth of bacteri

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- If $0 < b < 1$, then th

Remember:
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An exponential graph

$b > 1$

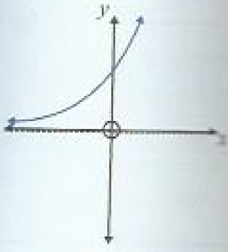


Figure 1.34

Activity 9

For each of the following:

a) Give the equation of the inverse in standard form.
b) If necessary, restrict the domain of the function so that its inverse will be a function.
c) On the same set of axes sketch the graphs of the function and its inverse.

1 $f(x) = -2x - 2$
2 $2x - y + 3 = 0$
3 $g(x) = \frac{1}{2}x^2$
4 $p(x) = -x^2$

Exponential graphs

In an exponential function the variable x is a power of a constant number. The standard formula of an exponential function is: $f(x) = b^x$, where $b > 0$ and $b \neq 1$.

Did you know?

Exponential graphs are used to represent situations of rapid growth, for example, the rapid growth of bacteria in culture.

- If $b > 1$, then the graph is an increasing function.
- If $0 < b < 1$, then the graph is a decreasing function.

Remember:
A function is increasing if the values of y increase as x increases.
A function is decreasing if the values of y decrease as x increases.

An exponential graph has a horizontal asymptote, but no vertical asymptote.

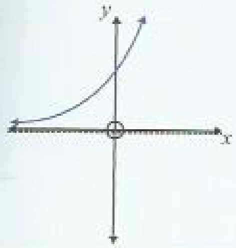
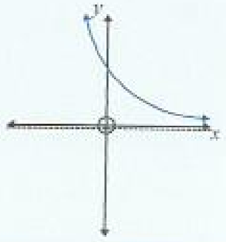
$y = b^x$	
$b > 1$	$0 < b < 1$
	

Figure 1.34

New word

asymptote: A line which a graph approaches but never reaches.

Worked example 10

1 Sketch the graphs of:

a) $y = 2^x$ b) $y = (\frac{1}{2})^x$

State the intercepts, domain, range, whether the graph is increasing or decreasing and the asymptote of each function.

Answer

1 Table of values:

x	−2	−1	0	1	2	3	4
y	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16

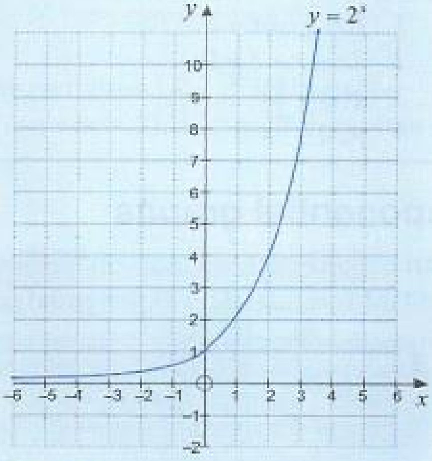


Figure 1.35

- The graph passes through the point (0, 1).
- The domain is the set of all real numbers.
- The range is $y > 0$.
- The graph is increasing.
- The x-axis is an asymptote to the graph as x approaches negative infinity.

2 Table of values:

x	−2	−1	0	1	2	3	4
y	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$

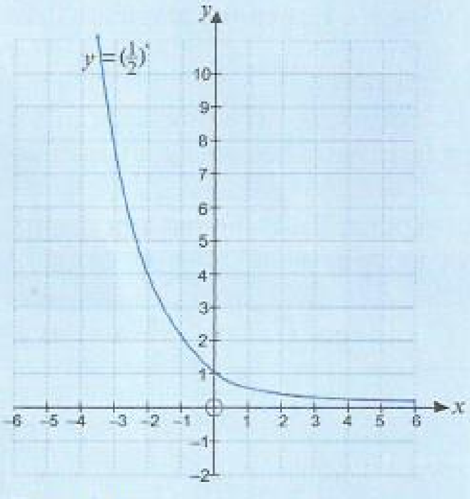


Figure 1.36

- The graph passes through the point (0, 1).
- The domain is the set of all real numbers.
- The range is $y > 0$.
- The graph is decreasing.
- The x-axis is an asymptote to the graph as x approaches positive infinity.

Activity 10

1 Sketch the graphs of the following on separate sets of axes:

a) $y = 3x$ b) $y = (\frac{1}{3})^x$ c) $y = 10x$ d) $y = (\frac{1}{10})^x$

Activity 10 (co

2 In 1995, there
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a) Write an ex
b) Draw up a t
c) How many
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$b \neq 1$ and $x > 0$.

We read this as "log

Worked exampl

1 Find the inverse
a) $y = 10^x$
2 Given $f(x) = 4^x$.
a) Sketch the gr
both graphs
b) State the inte
c) Label A, the
d) State the dom

Answers

1 a) $y = 10^x$
inverse: $x =$ 1
standard form
b) $y = 3^x$
inverse: $x = 3$
standard form
c) $y = (\frac{1}{2})^{2x}$
inverse: $x = ($
standard form

Activity 10 (continued)

2 In 1995, there were 285 people with cellphones in a city. After 1995 the number of people with cellphones increased by 75% each year.
- a) Write an exponential formula for this relationship.
- b) Draw up a table of some chosen values and draw a graph of the function.
- c) How many people had cellphones in this city by 2014? (Round off your answer to the nearest person.)

The inverse of the exponential function

Consider the exponential function $y = b^x$. To determine the inverse of the exponential function, we interchange the x- and y-variables to get $x = b^y$.

But now we have the problem that we don't have a method of writing y as the subject of the formula, because y is the exponent.

The logarithmic function allows us to rewrite the expression $x = b^y$ with y as the subject of the formula: $y = \log_b x$

If $x = b^y$, then $y = \log_b x$, where $b > 0$, $b \neq 1$ and $x > 0$.

We read this as "log x to the base b".

New word

logarithm: The logarithm of a number is the exponent to which a base must be raised to produce the number.

Worked example 11

1 Find the inverses of these functions and write them in standard form.

a) $y = 10^x$ b) $y = 3^x$ c) $y = (\frac{1}{2})^{2x}$

2 Given $f(x) = 4^x$.
- a) Sketch the graphs of $f(x)$ and $f^{-1}(x)$ on the same system of axes. Label both graphs clearly.
- b) State the intercept(s) for each graph.
- c) Label A, the point of intersection of $f(x)$ and $f^{-1}(x)$.
- d) State the domain, range and asymptote(s) of each function.

Answers

1 a) $y = 10^x$

inverse: $x = 10^y$

standard form: $y = \log_{10} x$

b) $y = 3^x$

inverse: $x = 3^y$

standard form: $y = \log_3 x$

c) $y = (\frac{1}{2})^{2x}$

inverse: $x = (\frac{1}{2})^{2y}$

standard form: $y = \frac{1}{2}\log_{\frac{1}{2}} x$

Worked example 11 (continued)

2 $f(x) = 4^x$

$f^{-1}(x) = \log_4 x$

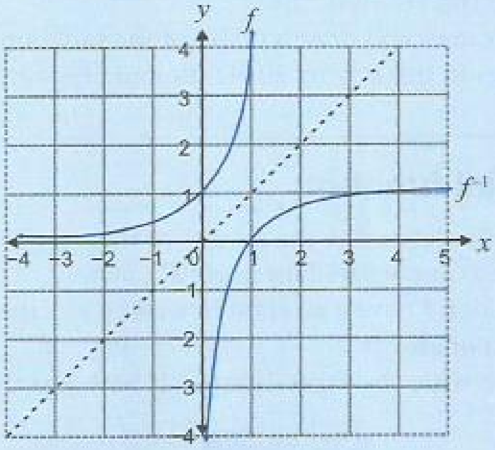

Figure 1.37

Did you know?

A **logarithmic scale** is a non-linear scale that we use when we want to represent a wide range of values. It is based on orders of magnitude to the base 10.

Some examples include the strength of earthquakes (the Richter scale), the loudness of sound (the decibel scale), and the pH of solutions.

We often leave out the number 10 when we use common logarithms with a base of 10. So log 100 means $\log_{10}100$ and is equal to 2.

Activity 11

1 Find the inverses of the following functions and write them in standard form.

a) $y = \log_3 x$ b) $y = 5^x$ c) $y = 10^x$

d) $y = 100^x$ e) $y = a^x$ f) $y = \log_{\frac{1}{4}} x$

2 Write down the values of the following logarithms:

a) log 1 000 b) $\log_3 27$ c) log 0.00001

d) $\log_4 16$ e) log 0.01 f) log 10

g) log 1 h) $\log_5 125$

3 Given $f(x) = (\frac{1}{3})^x$

a) Sketch the graphs of $f(x)$ and $f^{-1}(x)$ on the same system of axes. Label both graphs clearly.

b) State the intercept(s) for each graph.

c) Label A, the point of intersection of $f(x)$ and $f^{-1}(x)$.

d) State the domain, range and asymptote(s) of each function.

4 Determine the equation of the graph formed if $y = \log_5 x$ is reflected about:

a) the x-axis b) the y-axis c) the line $y = x$.

5 Determine the value of a of the function if:

a) $y = \log_a x$ passes through $(\frac{1}{4}, -2)$

b) $y = a^x$ passes through (1, 0.2)

Activity 11 (co

*6 **[Extension]** Th

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a) Write a form

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(i) 2×10^{-4}

(ii) 1

(iii) 0.004

d) Why do you

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c) $y = 10^x$
f) $y = \log_{\frac{1}{4}} x$

c) log 0.00001
f) log 10

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c) the line $y = x$.

Activity 11 (continued)

*6 **[Extension]** The pH scale measures how acidic or alkaline a solution is. The pH scale ranges from 0 to 14. A pH of 7 is neutral. A pH less than 7 is acidic. A pH greater than 7 is alkaline.

The pH scale is a negative logarithmic scale. So a pH of 3 refers to a hydrogen concentration of 10^{-3}.

a) Write a formula that relates the pH (P) to the concentration of hydrogen (H) in a solution.

b) The pH values of solutions can range from 0 to 14. Plot a graph of the pH scale on graph paper.

c) Use your graph to read off the pH where the concentration of hydrogen is equal to:

(i) 2×10^{-4}
(ii) 1
(iii) 0.004

d) Why do you think a logarithmic scale is used for pH values?

TOPIC

1

Summary, revision and assessment

Summary

- The derivative is the instantaneous rate of change of a function, or the gradient, at a point. This is equal to the gradient of a tangent at that point.
- The standard form of a cubic function is $f(x) = ax^3 + bx^2 + cx + d$.
- To sketch cubic graphs:
  - The a value of the formula written in standard form gives the shape of the graph.
  - If the equation can be factorised, we can find the zeros of the graph by using the zero product rule.
  - We find the stationary points by finding the points where the derivative of the function (the gradient at a point) is equal to zero. The stationary points are either the local maximum and minimum points, or the point of inflection.
  - The point of inflection of the graph is the point where the second derivative of the function is equal to zero.
- We can estimate the solutions of a cubic graph by plotting the graph accurately using the critical points and plotting some other points.
- We can estimate the area between a curve and the x-axis by drawing rectangles or trapeziums of equal widths to fill the space underneath a curve, and finding the sum of the area.
- Functions can be one-to-one relations or many-to-one relations. A many-to-one relation associates two or more values of the independent (input) variable with a single value of the dependent (output) variable.
- Given a function $f(x)$, we can determine the equation of the inverse $f^{-1}(x)$ by:
  - interchanging the x- and y-values
  - making y the subject of the equation
  - expressing the new equation in function notation.
- The domain of the function needs to be restricted in some cases to ensure that the inverse is a function.
- The inverse of the exponential function $f(x) = b^x$; $(b > 0, b \neq 1)$ is the logarithmic function $f^{-1}(x) = \log_b x$.

Revision exercises (Remedial)

1 Write down the general forms of the quadratic function and the cubic function.

2 Give the zeros of the following equations.

a) $y = x^2 - 4$ b) $y = (x + 1)(x^2 - 9)$

3 Determine the value of a in $f(x) = a^x$ if the function passes through the point:

a) $(1, 4)$ b) $(3, 8)$ c) $(2, 25)$ d) $(6, 64)$

Summary, revis

4 Determine the

a) $(8, 3)$

Revision exer

1 Use differentiati

minimum turni

a) $f(x) = x^2 + 4x$

b) $f(x) = x^2 - 6x$

c) $f(x) = 2x^2 - 4$

d) $f(x) = 5 - 4x$

e) $f(x) = 4 - 8x$

f) $f(x) = 7 - x -$

2 Sketch the graph

a) $f(x) = x^3 - 3x$

b) $f(x) = x^3 + x^2$

c) $f(x) = x^3 + 2x$

d) $f(x) = \frac{1}{3}x^3 - \frac{1}{2}$

3 Draw accurate gr

a) $f(x) = x^3 - \frac{1}{2}x$

b) $f(x) = x^3 - x^2$

4 Use the areas of

curve and the x-

a) $y = 2x^2 + 3$, fr

b) $y = x^2 + 2$, fro

5 Use the areas of

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a) $y = 2x^2 + 3$, fr

b) $y = \frac{2}{x} + x$, fro

6 Determine the e

a) $(8, 3)$

b) $(3, -1)$

c) $(\frac{1}{2}, -1)$

7 Draw sketch grap

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6, 64)

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4 Determine the value of a in $f(x) = \log_a x$ if it passes through the point:
a) (8, 3) b) (125, 3) c) (100, 2) d) (49, 2)

Revision exercises

1 Use differentiation to state whether the following have a maximum or minimum turning point. Find the coordinates of the turning point.
a) $f(x) = x^2 + 4x - 5$
b) $f(x) = x^2 - 6x + 2$
c) $f(x) = 2x^2 - 4x + 7$
d) $f(x) = 5 - 4x - x^2$
e) $f(x) = 4 - 8x - x^2$
f) $f(x) = 7 - x - 2x^2$

2 Sketch the graphs of:
a) $f(x) = x^3 - 3x^2 - 4x$
b) $f(x) = x^3 + x^2 - 5x$
c) $f(x) = x^3 + 2x^2 - 4x - 5$
d) $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x + 1$

3 Draw accurate graphs of the following to find the zeros.
a) $f(x) = x^3 - \frac{1}{2}x^2 + 2x + 3$
b) $f(x) = x^3 - x^2 - 5x + 2$

4 Use the areas of rectangles method to estimate the area between the function curve and the x-axis within the given interval.
a) $y = 2x^2 + 3$, from $x = 1$ to $x = 2$
b) $y = x^2 + 2$, from $x = 2$ to $x = 3$

5 Use the areas of trapeziums method to estimate the area between the function curve and the x-axis within the given interval.
a) $y = 2x^2 + 3$, from $x = 0$ to $x = 2$
b) $y = \frac{2}{x} + x$, from $x = 2$ to $x = 3$

6 Determine the equation of $y = \log_a x$ if it passes through the point:
a) (8, 3)
b) (3, −1)
c) $(\frac{1}{2}, -1)$

7 Draw sketch graphs of each of the functions in Question 6.

Summary, revision and assessment (continued)

Assessment exercises

1 Determine the coordinates and nature of the turning points of the function $f(x) = x^3 - x^2 - 5x + 4$. Sketch the curve.

2 Sketch the graphs of:
  a) $f(x) = (x - 1)^3$
  b) $g(x) = (x + 1)(x^2 - 4x + 5)$

3 Sketch the graph of $f(x) = ax^3 + bx^2 + cx + d$, given the following information:
- $f(4) = 0$
- $f'(2) = 0$
- $f'(2) = 8$; $f(0) = 16$
- $f'(x) < 0$ if $x < 2$
- $f'(x) < 0$ if $x > 2$

4 Builders need to fill an arch which is 2 m wide and 3 m high.
  a) Determine the equation of the parabola (assuming that it begins at $x = 0$).
  b) Estimate the area under the arch, using the sum of areas of rectangles method.
  c) Estimate the area under the arch, using the sum of areas of trapeziums method.

5 The point A(–2, 4) lies on the graph of $f(x) = a^x$.
  a) What is the formula of $f(x)$?
  b) Write the formula of $f^{-1}(x)$ in standard form.
  c) Sketch the graphs of $f(x)$ and $f^{-1}(x)$.

Assessment exercises (extension)

1 A population of 24 cockroaches doubles every month.
  a) Determine a formula that describes the growth of the population.
  b) Calculate how long it will take for the cockroach population to reach 100 000.

2 The International Space Agency has landed a robotic explorer on a comet. Some probes are extended from the lander's body to conduct various tests. To demonstrate the force of gravity on this comet, the lander launches a ball directly upwards at 150 m/s. The acceleration due to gravity a on this comet is equal to 2 m/s². At what time will it fall back onto the comet?
Use the quadratic equation $s = at^2 + vt$, where s is the displacement, t is the total time taken, v is the initial velocity and a is the acceleration due to gravity. When the ball is back in its original position, the displacement is 0, so $s = 0$.

TOPIC
2
Lin

Sub-topic

Linear programming

Starter activity

1 a) Give the equati

b) Give the coordi

2 Solve the following
also show them on

a) $x - 3 > -7$

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TOPIC 2

Linear programming

Sub-topic	Specific Outcomes
Linear programming	• Draw graphs of linear equations and inequations in one and two variables (as a recap). • Shade the wanted and unwanted regions. • Describe the wanted or unwanted regions. • Determine maximum and minimum values. • Use the search line to determine the maximum and minimum values. • Apply knowledge of linear programming in real life.

Starter activity

1 a) Give the equations of lines A and B.

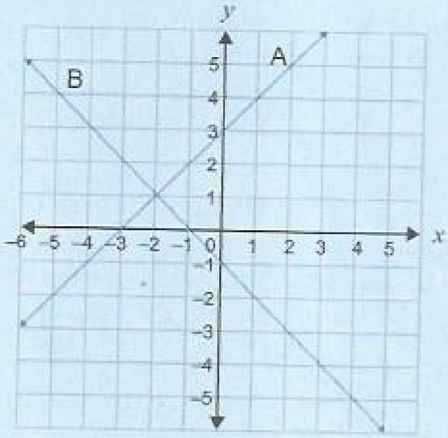

Figure 2.1

b) Give the coordinates of the point of intersection of A and B.

2 Solve the following inequations. Give the solutions in interval form and also show them on a number line.

a) $x - 3 > -7$ b) $-1 \leq 2 - x < 7$

3 A manufacturer of school shirts has a fixed monthly cost of K6 000 and it costs him K40 to make each shirt. Write a linear equation to represent this. Use c to represent the total monthly cost and n to represent the number of shirts.

SUB-TOPIC 1 Linear programming

Linear equations

A linear equation is an equation where the highest power of the variable is 1.
Example: $5x + 3 = 4$

Solving equations and inequations in one variable

To solve an equation for one variable x in an equation, you need to find the value of x that will make the equation true. To do this, rewrite the equation so that x is on its own on one side of the equation. This involves working with additive and multiplicative inverses.

Worked example 1

1 Solve for x.

a) $3(1 - x) - (x + 2) = 9$ b) $\frac{x+2}{3} - x = \frac{1}{2}$ c) $\frac{2x+1}{3} - \frac{x-2}{4} = 5$

Answers

1 a) $3(1 - x) - (x + 2) = 9$

$3 - 3x - x - 2 = 9$ Multiply out.

$-4x + 1 = 9$ Simplify.

$-4x = 8$ Subtract 1 from both sides.

$x = -2$ Divide both sides by −4 to get x on its own.

b) $\frac{x+2}{3} - x = \frac{1}{2}$

$2(x + 2) - 6x = 3$ Multiply through by the LCD of 6.

$2x + 4 - 6x = 3$

$-4x = -1$ Simplify and subtract 4 from both sides.

$x = \frac{1}{4}$ Divide both sides by −4 to get x on its own.

c) $\frac{2x+1}{3} - \frac{x-2}{4} = 5$

$4(2x + 1) - 3(x - 2) = 60$ Multiply through by the LCD of 12.

$8x + 4 - 3x + 6 = 60$

$5x = 50$ Simplify and subtract 10 from both sides.

$x = 10$ Divide both sides by 5 to get x on its own.

An inequation is similar to an equation but the two expressions are related by one of these symbols: greater than (>); less than (<); greater than or equal to (≥); or less than or equal to (≤).

You solve inequations in the same way that you solve equations. However, if you multiply or divide both sides of an inequation by a negative number, the direction of the inequation symbol must change.

The solution to an inequation is usually a range of values rather than a single value.

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Worked example 2

1 Solve for x and show the solutions on a number line.

a) $x - 3 < 2$
b) $4x + 6 \leq 3x - 5$
c) $-3x + 5 \geq 14$
d) $\frac{x-4}{3} \leq 2(x + 1) < 1 + 3x$

Answers

1 a) $x - 3 < 2$

$x < 2 + 3$ Add 3 to both sides.

$x < 5$

b) $4x + 6 \leq 3x - 5$

$4x - 3x \leq -5 - 6$

$x \leq -11$

c) $-3x + 5 \geq 14$

$-3x \geq 14 - 5$

$-3x \geq 9$

$x \leq -3$ Change the direction of the sign, because we divide by −3.

d) $\frac{x-4}{3} \leq 2(x + 1) < 1 + 3x$

This inequation has two parts, which can be solved separately.

$\frac{x-4}{3} \leq 2(x + 1)$	$2(x + 1) < 1 + 3x$
$x - 4 \leq 6x + 6$	$2x + 2 < 1 + 3x$
$x - 6x \leq 6 + 4$	$-x < -1$
$-5x \leq 10$	$x > 1$
$x \geq -2$	

The solution is where $x \geq -2$ and where $x > 1$ ∴ $x > 1$

Activity 1

1 Solve for x.

a) $24 - 8x = 2 + 3x$

b) $2(x - 1) = 5 - 2(x + 4)$

c) $7 - 2(x + 3) = 2 - 5(x - 1)$

d) $4(x + 1) = 6(x - 2) + 8$

e) $\frac{x+1}{x} - \frac{2}{3} = \frac{1}{2}$

f) $\frac{x-3}{2} - 1\frac{1}{2} = 2x$

2 Solve for x.

a) $2x + 2 \leq 1$

b) $\frac{4}{3}x - 6 < 7x + 2$

c) $2 - x \geq 2(3x + 1)$

d) $-1 \leq 2 - x < 7$

e) $2x + 3 \leq 7$

f) $\frac{7x-1}{3} \geq 2$

3 Kachana has K5 000 in a savings account at the beginning of the year. He wants to always keep a minimum balance of K2 000 in his account. He withdraws K150 from his account each week.

a) Write an inequation for this situation using the variable x for the number of weeks.

b) For how many weeks can he continue to withdraw money from his account?

4 A taxi charges a flat rate of K5 plus an additional K1.50 per 5 km travelled. How far can you travel if you have a maximum of: a) K15 b) K50?

5 Ganizani and Chawezi play in the same soccer team. Last Saturday Chawezi scored 3 more goals than Ganizani. Together they scored fewer than 12 goals. What is the possible number of goals that Chawezi scored?

6 A school has K45 000 to spend on computers. They need to buy a printer which will cost K9 000. How many computers can they buy if each computer costs K5 000?

*7 **[Extension]** The velocity in metres per second (m/s) of a stone thrown straight up into the air is given by the equation $v = 50 - 12t$, where t is the time in seconds. The time at which the stone is thrown is $t = 0$.

a) At what times will the velocity be between 10 m/s and 15 m/s?

b) At what times will the velocity be less than 50 m/s?

*8 **[Extension]** If you invest K10 000 in a savings account which yields 2.5% interest each year, at the end of how many years will the amount in the savings account be greater than K20 000?

Solving equations and inequations in two variables

A linear function has the form $y = ax + b$, where the constant a represents the gradient of the function and b represents the y-intercept.

The solution to **one** linear equation in two variables is any point on the graph of that equation. This is because any point on the graph will satisfy the equation. To find definite values for the two variables, we need to have **two** linear equations. We can solve linear equations in two variables in the following ways:

$- 2(x + 4)$
$(x - 2) + 8$
$2x$

$+ 2$
7

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- A graphical method. Here we find that the point where the two graphs intersect is the only point that satisfies both equations, so that point is the solution.
- An algebraic method. Here we make use of substitution and elimination techniques to find the solution.

New words

substitution: solving a system of equations by solving one of the equations for one of the variables, and then putting this value back into the other equation.
elimination: solving a system of equations by eliminating (getting rid of) one of the variables by doing operations on the equations and combining them.

Worked example 3

1 Plot the graphs of each pair of equations. Then solve the pair of equations for x and y using an algebraic method and compare the solutions to the graphical solutions.

a) $y = -x + 1$ and $y = -\frac{2}{3}x$ b) $y = -2x - 2$ and $y = \frac{1}{4}x + 2$

Answers

1 a) Graphical method:

Both equations are in standard form (also called "gradient–intercept" form) so we can plot the graphs directly from the equations.

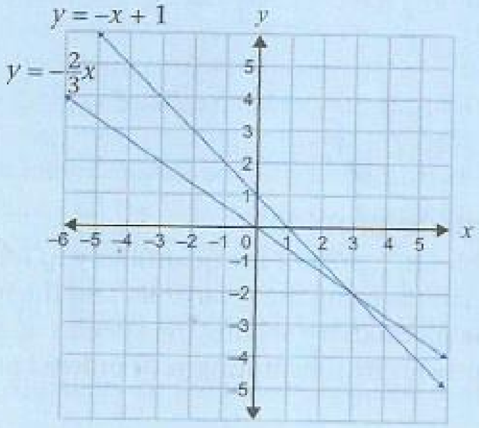

Figure 2.2

The two lines appear to intersect at the point (3, –2).

Algebraic method (using substitution):

Substitute $y = -\frac{2}{3}x$ into $y = -x + 1$.

$-\frac{2}{3}x = -x + 1$

$\frac{1}{3}x = 1$

$\therefore\ x = 3$

Substitute $x = 3$ into $y = -x + 1$

$\therefore y = -2$

$\therefore$ the solution is (3, –2).

Note

The graphical method is accurate for whole number solutions, but it is sometimes more difficult to get accurate solutions for fractions.

Worked example 3 (continued)

b) Graphical method:

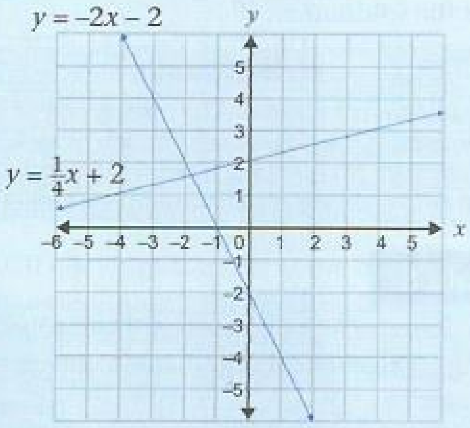

Figure 2.3

The point where the two graphs intersect appears to be at about $(-1\frac{3}{4}, 1\frac{3}{4})$

Algebraic method (using elimination and substitution):

$y = -2x - 2$ ①

$y = \frac{1}{4}x + 2$ ② Subtract ① – ② to eliminate y

$0 = -2\frac{1}{4}x - 4$

$x = 4 \times -\frac{4}{9} = -\frac{16}{9} = -1\frac{7}{9}$

Substitute $x = -1\frac{7}{9}$ into ①: $y = -2(-\frac{16}{9}) - 2 = \frac{32}{9} - 2 = 1\frac{5}{9}$

$\therefore$ the solution is $(-1\frac{7}{9}, 1\frac{5}{9})$

How to solve real life problems

Step 1: Choose variables to represent the quantities you want to find.

Step 2: Write two equations using these variables and the information given.

Step 3: Use one of the methods for solving the systems of equations.

Step 4: Check your answers by substituting your ordered pair into the original equations.

Worked example 4

1 A waiter at a fast-food shop gives one group of people a bill for K140 for two toasted sandwiches and three burgers. Another group get a bill for K160 for four sandwiches and two burgers. Calculate the cost of one toasted sandwich and the cost of one burger.

Worked exam

Answer

1 Let the numbe
Write the equa

$2t + 3b = 140$

$4t + 2b = 160$

$\underline{4t + 6b = 280}$

$-4b = -120$

$b = 30$

Substitute $b =$ 3

$2t + 3(30) = 14$

$2t = 50$

$t = 25$

So burgers cost

Activity 2

1 Draw graphs o
an algebraic m

a) $y = x + 9$ an

c) $x - 3y = 6$ a

2 Solve the follo
solving them u

a) A school ha
15 vehicles
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b) The total co
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*3 **[Extension]** W
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Linear inequatio

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Example: $y \leq -2x$

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Worked example 4 (continued)

Answer

1 Let the number of toasted sandwiches be t and the number of burgers be b.
Write the equations and solve by elimination.

$2t + 3b = 140$ ①
$4t + 2b = 160$ ②
$\underline{4t + 6b = 280}$ ③ Multiply ① by 2 to get equation ③.
$-4b = -120$ Subtract ③ from (2).
$b = 30$

Substitute $b = 30$ back into ①:

$2t + 3(30) = 140$
$2t = 50$
$t = 25$

So burgers cost K30 each and toasted sandwiches cost K25 each.

Activity 2

1 Draw graphs of the following pairs of linear equations and solve them using an algebraic method.
  - a) $y = x + 9$ and $y = 2x - 3$
  - b) $y = -\frac{1}{4}x - 2$ and $y = \frac{3}{2}x + 3$
  - c) $x - 3y = 6$ and $x + y + 6 = 0$
  - d) $y = x - 6$ and $y = -x + 2$

2 Solve the following problems by first writing two equations and then solving them using an algebraic method.
  - a) A school has 231 learners who all go out on a school outing. There are 15 vehicles in total: minibuses and bigger buses. The minibuses can each seat 7 people, and the bigger buses can carry 25 learners each. How many of each type of vehicle did they use?
  - b) The total cost of 12 apples and 8 mangos is K84, while the total cost of 8 apples and 12 mangos is K96. What is the cost of each apple and mango?

*3 **[Extension]** Write equations for the following problem and solve them using substitution.
The sum of two numbers is 17 and the sum of their squares is 145. What are the numbers?

Linear inequations (inequalities)

You have solved inequations in one variable, and plotted the solution on a number line, for example $x \leq 3$.

For an inequation in two variables, we write it in the standard form $y \leq mx + b$, like the equation for a straight line. This makes it easier to plot the graph.

Example: $y \leq -2x + 3$.

How to graph an inequation

Step 1: Graph the boundary line of the inequation.

- Plot the line as a broken line if the boundary line is excluded (if a $<$ or $>$ sign is used).
- Plot the line as a solid line if the boundary line is included (if a $\leq$ or $\geq$ sign is used).

Step 2: Choose a simple point which is not on the line and substitute it into the inequation.

- If the point makes the inequation true, then it is in the solution region.
- If the point does not make the inequation true, then the solution region is on the other side of the line.

Step 3: Shade the solution region.

Note

Remember that the set of all points (x, y) that satisfies the equation $y = -2x + 3$ are all the points that lie on the line. So the solution to $y \leq -2x + 3$, is the set of all the points that lie on the line or below it.

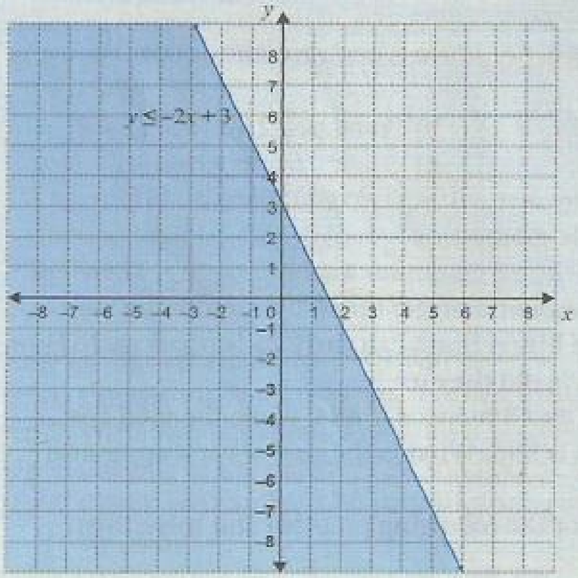

Figure 2.4

The area under the line is shaded and the boundary line is solid. This indicates that the line and the points below it are included in the solution.

To test that the correct area is shaded, substitute some sample points into the inequation: for example (0, 0) and (4, 1).

(0, 0)	(4, 1)
$y \leq -2x + 3$	$y \leq -2x + 3$
$0 \leq -2(0) + 3$	$1 \leq -2(4) + 3$
$0 \leq 3$	$1 \leq -5$
This is a true statement so the correct area is shaded.	This is not a true statement so this point is not in the shaded region.

Worked example

1 Graph the solutio
2 Draw the graphs

Answers

1 First write the ine
$2x - 3y < 6$
$-3y < -2x + 6$
$y > (\frac{2}{3})x - 2$
The direction of t
due to dividing by
The line is broken
included in the so
Test point: (1, 1)
$2x - 3y < 6$
$2(1) - 3(1) < 6$
$2 - 3 < 6$
$-1 < 6$
This is a true state
region above the l
be shaded.

2 $y > x + 1$ and $y < x$

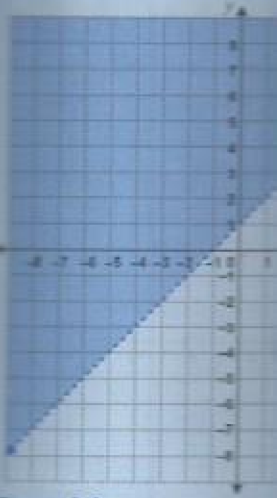

Figure 2.6

Test point: (0, 3)
$y > x + 1$
$3 > 0 + 1$
$3 > 1$
This is a true statemen
region above the line

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of all the points that lie

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gion.

Worked example 5

1 Graph the solution to $2x - 3y < 6$.

2 Draw the graphs of $y > x + 1$ and $y < x + 1$ on separate sets of axes.

Answers

1 First write the inequation in standard form.

$$2x - 3y < 6$$
$$-3y < -2x + 6$$
$$y > \left(\tfrac{2}{3}\right)x - 2$$

The direction of the sign changes due to dividing by −3.

The line is broken, as it is not included in the solution.

Test point: (1, 1)

$$2x - 3y < 6$$
$$2(1) - 3(1) < 6$$
$$2 - 3 < 6$$
$$-1 < 6$$

This is a true statement so the region above the line must be shaded.

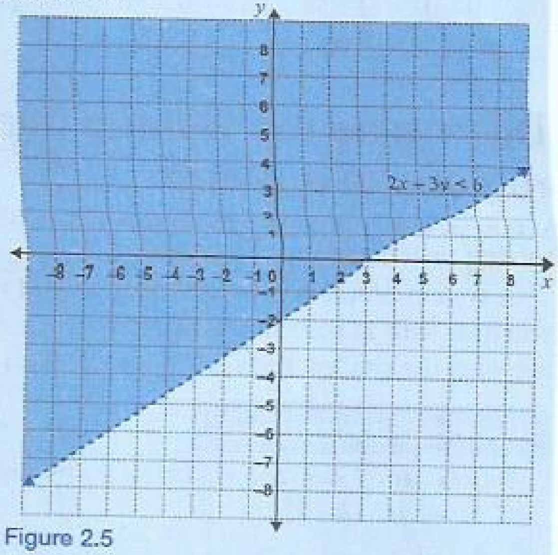

Figure 2.5

2 $y > x + 1$ and $y < x + 1$

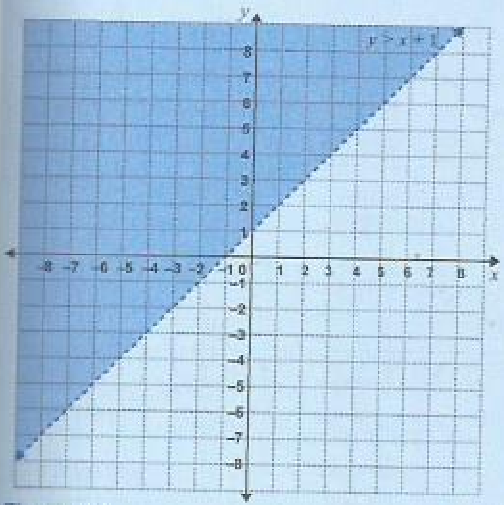

Figure 2.6

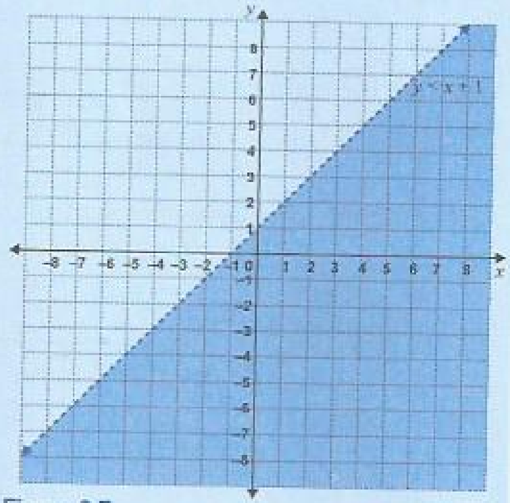

Figure 2.7

Test point: (0, 3)

$y > x + 1$

$3 > 0 + 1$

$3 > 1$

This is a true statement so the region above the line is shaded.

Test point: (0, −3)

$y < x + 1$

$-3 < 0 + 1$

$-3 < 0$

This is a true statement so the region below the line is shaded.

Activity 3

1 a) Plot the graph of $x \geq 2$ on a number line.

b) Plot the graph of $x \geq 2$ on the Cartesian plane.

2 Sketch the following graphs.

a) $y > x$ and $y < x$

b) $3x - 2y < 5$

c) $y - x \leq 2$

New word

Cartesian plane (coordinate plane): flat area containing the x-axis and the y-axis

Determine the wanted (feasible) and unwanted regions of a graph

When you know how to graph individual linear inequations, you can move on to solving systems of linear inequations.

A system of linear inequations is a set of two or more linear inequations that you deal with at the same time. This will result in a few shaded areas. The overlap between the shaded areas is the area that satisfies all of them.

Worked example 6

1 Solve the following system of inequations.

$2x - 3y \leq 12$

$x + 5y \leq 20$

$x \geq 0$

Answer

Write the inequations in standard form.

$y \geq (\frac{2}{3})x - 4$

$y \leq (-\frac{1}{5})x + 4$

$x \geq 0$

Sketch the linear graphs.

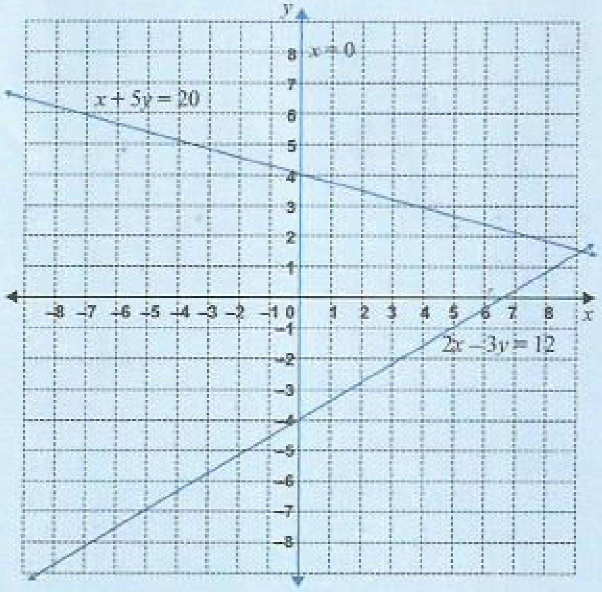

Figure 2.8

Worked exam

The graph of $x = 0$ i

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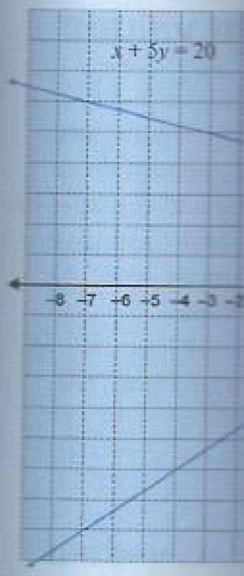

Figure 2.9

Last, show the regio

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This is the **wanted** or

unwanted region.

In linear programm

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Activity 4

1 Sketch the follow

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a) $x + y \leq 2$

$3x - 2y + 6 <$

$x > -1$

$y \geq 0$

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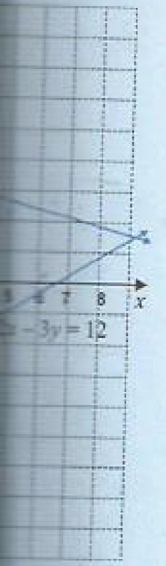

Worked example 6 (continued)

The graph of $x = 0$ is the y-axis.

Then shade in the solution region for each graph.

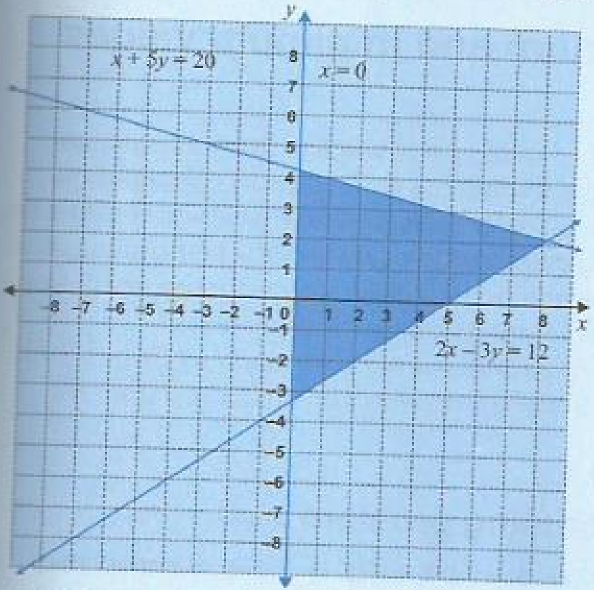

Figure 2.9

Last, show the region where all three of the inequations are satisfied. This is indicated by the overlapping region between all three lines.

In the system of equations in Worked example 6, the three inequations form boundary lines for a triangular region which is the solution area for all of them. This is the **wanted** or **feasibility** region, and the rest of the coordinate plane is the **unwanted** region.

In linear programming, however, we show the feasible region as an unshaded area and so the unwanted region is the shaded area.

New words

feasible (wanted) region: the solution area on a graph of one or more inequations

not feasible (unwanted) region: the rest of the coordinate plane which is not occupied by the solution area on a graph of one or more inequations

Activity 4

1 Sketch the following system of inequations and leave the feasible region **unshaded**.

a) $x + y \leq 2$
$3x - 2y + 6 < 0$
$x > -1$
$y \geq 0$

b) $3x - y \leq 6$
$-2 < x < 2$

c) $x + 2y < 1$
$x \leq 2$
$y > -1$

d) $6y + x \geq 11$
$11 > 2x + y$
$3x - y \leq 6$

e) $2x + y > -1$
$2y - x + 2 \leq 0$

f) $y - 3 \leq 0$
$2x + y + 4 \geq 0$
$x - y - 1 \leq 0$

Determine the maximum and minimum values for a problem

Linear programming is the process of taking a few different linear inequations relating to a particular situation, and finding the optimal (best) value in those circumstances. For example, in the manufacturing industry we could consider the materials and labour available for a particular job, and find the best combination of the number of items that can be produced to make the maximum profit. Note that not all mathematically correct solutions are practical solutions.

So far, we have found a region of possible solutions by graphing inequations. The inequations that we graphed in the previous section are formed from the **constraints** of the situation.

Solving a linear programming problem

Linear programming problems always have information about quantities that need to be optimised (maximised or minimised), depending on a few conditions (constraints). To solve this kind of problem we need to identify and write down an equation using the information given. We call this **optimisation** equation the **objective function**.

We then need to find the coordinates of the vertices of the feasible region. The maximum and the minimum values are always found at the coordinates of these vertices. Finally we need to substitute the values of these **vertices** into the objective function to find the maximum or minimum value.

New words

constraints: the conditions that must be satisfied in an optimisation problem
optimise: to find the best solution
objective function: the equation we use to determine the optimal (maximum or minimum) solution
vertices (plural of vertex): the corner points of a geometrical figure

How to solve a linear programming problem

Step 1: Choose the variables.
Step 2: Write the objective function.
Step 3: Determine the constraint inequations and arrange into standard form.
Step 4: Draw graphs of the inequations and indicate the feasible region.
Step 5: Find the coordinates of the vertices and substitute the values into the objective function.
Step 6: Write down the answer.

$-3 \le 0$
$2x + y + 4 \ge 0$
$x - y - 1 \le 0$

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Worked example 7

1 Find the maximum and minimum value of $z = 3x + 4y$ with the following constraints:
$x + 2y \le 14$
$3x - y \ge 0$
$x - y \le 2$

2 Mrs Lupunga has a business where she designs and makes copper jewellery. She makes a profit of K25 from each bracelet and K30 from each necklace. The business can make a maximum of 500 of each item every month, and they can manufacture up to 800 items per month. She must make at least one-third as many bracelets as necklaces. How many of each item should she make to get the most profit?

Answers

1 Objective function: $z = 3x + 4y$.
Constraints (inequations in standard form):
$y \le -\frac{1}{2}x + 7$ ①
$y \le 3x$ ②
$y \ge x - 2$ ③
Graph the inequations. Shade the regions that are not feasible and leave the feasible area unshaded. From the graph we can see that the vertices are (2, 6), (6, 4) and (–1, –3).. (Sometimes we may need to use algebraic elimination to find the solution points.)
Substitute these values into the objective function $z = 3x + 4y$.

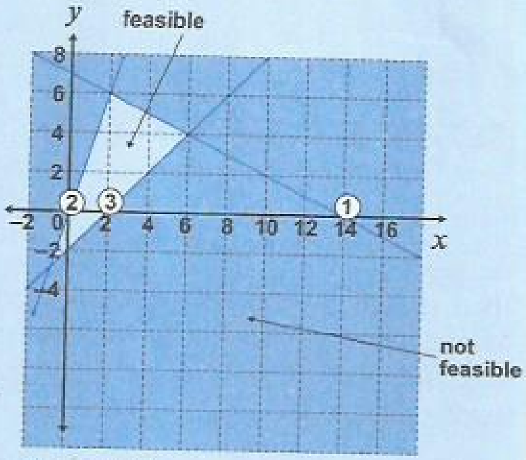

Figure 2.10

(2, 6): $z = 3(2) + 4(6) = 6 + 24 = 30$
(6, 4): $z = 3(6) + 4(4) = 18 + 16 = 34$
(–1, –3): $z = 3(-1) + 4(-3) = -3 - 12 = -15$
The maximum value of z is 34 and occurs at (6, 4).
The minimum value of z is –15 and occurs at (–1, –3).

2 Variables: x = number of bracelets (in hundreds)
y = number of necklaces (in hundreds)
Objective function: Profit = $2\,500x + 3\,000y$
Constraints (in standard form inequations):
$x \le 5$ ①
$y \le 5$ ②
$x + y \le 8 \therefore y \le -x + 8$ ③
$x \ge \frac{1}{3}y \therefore y \le 3x$ ④

Note
We multiply the price by 100 because each variable refers to 100 pieces of jewellery.

Worked example 7 (continued)

The problem also states that she must make at least one-third as many bracelets as necklaces so neither x nor y can be equal to zero. We can write two more constraints. (This restricts us to the first quadrant, not including the axes.)

$x > 0$ ⑤
$y > 0$ ⑥

Graph the inequations. Shade the regions that are not required. The feasible region is the part that remains unshaded.

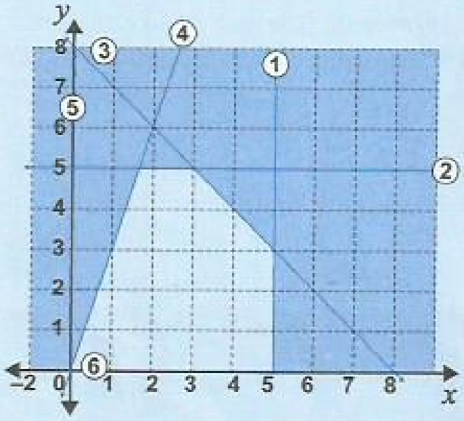

Figure 2.11

Vertices:

There are three vertices of the feasibility region that we need to consider.

Graphs ① and ③: $x = 5$, $y = -x + 8$, so $(x, y) = (5, 3)$
Graphs ② and ③: $y = 5$, $y = -x + 8$, so $(x, y) = (3, 5)$
Graphs ② and ④: $y = 5$, $y = 3x$, so $(x, y) = (\frac{5}{3}, 5)$

For the other vertices of the feasibility region:

- Considering the point where graphs ⑤ and ⑥ intersect: we have the point (0, 0) and both $x > 0$ and $y > 0$, so we don't consider this point.
- Also the point where graphs ① and ⑤ intersect does not exist, as $y > 0$.

Substitute the values into the objective function:

(5, 3): Profit $= 2\ 500(5) + 3\ 000(3) = 21\ 500$
(3, 5): Profit $= 2\ 500(3) + 3\ 000(5) = 22\ 500$
$(\frac{5}{3}, 5)$: Profit $= 2\ 500(\frac{5}{3}) + 3\ 000(5) = 19\ 167$

The point that maximises the profit of K22 500 is $(x, y) = (3, 5)$.
So Mrs Lupunga's business should make 300 bracelets and 500 necklaces per month.

Activity 5

1 A company laur
to award at least
are calculators,
Let the number
a) Write down
b) The compan
down two m
c) The cost of n
down a cost
company of
d) Sketch the gr
combination
e) How many o
company?
f) How much w
2 A bakery produc
of chocolate cup
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should they bake
3 Mr Muchimba is
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Use a search minimum valu

So far we have found t
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We can also use a se
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search line: a moving li
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The feasible

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ecklaces

Activity 5

1 A company launches a countrywide advertising campaign. They would like to award at least 40 prizes with a total value of at most K2 000. The prizes are calculators, valued at K60 each, and pens, valued at K40 each.
Let the number of calculators be x and the number of pens be y.
  - a) Write down two constraints other than $x > 0$ and $y > 0$.
  - b) The company decides that there will be at least 10 of each prize. Write down two more inequations for these constraints.
  - c) The cost of manufacturing a calculator is K24 and a pen is K16. Write down a cost equation which can be used to calculate the cost (C) to the company of the calculators and pens. This is the objective function.
  - d) Sketch the graph of the feasibility region for all the possible combinations of calculators and pens.
  - e) How many of each prize will represent the cheapest option for the company?
  - f) How much will this combination of calculators and pens cost?

2 A bakery produces two types of cupcakes: chocolate and vanilla. Each batch of chocolate cupcakes requires 4 kg flour and $\frac{1}{2}$ kg butter. Each batch of vanilla cupcakes requires 2 kg flour and 1 kg butter. The bakery has 3 000 kg flour and 1 200 kg butter available to use. How many batches of each kind should they bake to maximise the profit?

3 Mr Muchimba is a farmer who grows maize and sorghum. He must plant at most 40 hectares of the two crops. He must spend at least K132 000. It costs him K12 000 to sow one hectare of maize and K6 000 to sow one hectare of sorghum. He needs to plant more maize than sorghum, but he must plant at most 18 hectares of sorghum. If the profit on maize is K800 per hectare and on sorghum K500 per hectare, what combination of the two crops should the farmer plant to make a maximum profit and what is this profit?

Use a search line to find the maximum and minimum values

So far we have found the maximum or minimum values by substituting the values of the vertices of the feasible region into the objective function.

We can also use a **search line** to find these values by turning the objective function into a linear function and plotting it on the same graph.

New word

search line: a moving line with the same gradient as the objective function which is drawn through the vertices of the feasible region to search for the optimal value.

Let's look back at Question 1 of Worked example 7.

Find the maximum and minimum value of $z = 3x + 4y$ with the following constraints:

$x + 2y \leq 14$
$3x - y \geq 0$
$x - y \leq 2$

The objective function is $z = 3x + 4y$. How would we write a linear function to represent this? It has three variables so we cannot plot the graph. However we can write the equation in standard form and determine the gradient, which is all we need for a search line.

Objective function: $z = 3x + 4y$

$$\therefore y = \frac{-3x + z}{4}$$
$$\therefore y = -\frac{3}{4}x + \frac{1}{4}z$$
$$\therefore \text{Gradient is } -\frac{3}{4}.$$

Now draw lines with gradients equal to $-\frac{3}{4}$ that pass through each vertex of the feasible region. Look at the dashed lines in the graph below.

To draw a line with a gradient of $-\frac{3}{4}$, remember that:

- the line slopes down from left to right, as it is negative
- the change in y-value is 3 units for every 4 units change in the x-value.

Gradient of search line is: $-\frac{3}{4}$

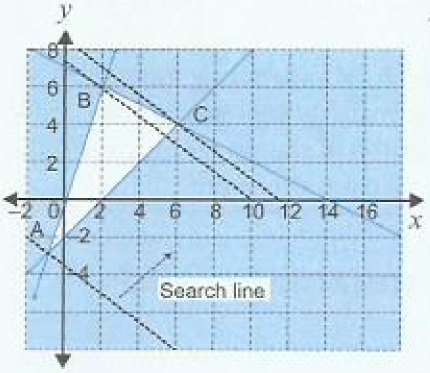

Figure 2.12

As the search line moves from left to right, with its constant gradient, the last vertex that it intersects is C(6, 4), so this vertex will give the maximum value of the objective function $z = 3x + 4y$.

Worked examp

Chitalu runs a smal
It takes him two ho
It takes two hours t
1 600 ml varnish in
least two small statu
If he makes K300
how many statues o
What is the most

Answer

Step 1: Choose the
Let the numl
Let the numl
Step 2: Write the ob
$p = 300x + 40$
Step 3: Write the co
He has 20 ho

He has 1 600

He must mak

Step 4: Plot the grap
Step 5: Find the grad
$p = 300x + 40$
Rearrange th
Gradient is –

As the search line m
right, the last vertex
at (8, 2), which is 8 s
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Substitute the coordi
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$p = 300x + 400y$
$p = 300(8) + 400(2) =$

So his maximum pro
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Worked example 8

Chitalu runs a small business making carvings for the tourist market in Lusaka. It takes him two hours to make a small wooden statue that needs 100 ml varnish. It takes two hours to carve a larger statue that needs 400 ml varnish. Chitalu has 1 600 ml varnish in stock. He can work for 20 hours. Chitalu has orders for at least two small statues and at least one large statue.

If he makes K300 profit on each small statue and K400 on each larger statue, how many statues of each type should he carve to make the most profit?

What is the most profit he can make?

Answer

Step 1: Choose the variables.

Let the number of large statues be y

Let the number of small statues be x

Step 2: Write the objective function: this is the profit equation

$p = 300x + 400y$

Step 3: Write the constraints making y the subject of the inequations.

He has 20 hours to do the work: $2y + 2x \leq 20$

$\therefore y \leq -x + 10$ ①

He has 1 600 ml varnish available: $100x + 400y \leq 1\ 600$

$\therefore y \leq -\frac{1}{4}x + 4$ ②

He must make at least two small statues and one large one:

$x \geq 2$ ③

$y \geq 1$ ④

Step 4: Plot the graphs.

Step 5: Find the gradient of the objective function for the search line.

$p = 300x + 400y$

Rearrange the equation: $y = -\frac{300}{400}x + \frac{p}{400}$

Gradient is $-\frac{3}{4}$

As the search line moves from left to right, the last vertex that it intersects is at (8, 2), which is 8 small statues and 2 large statues.

Substitute the coordinates (8, 2) into the objective function.

$p = 300x + 400y$

$p = 300(8) + 400(2) = 2\ 400 + 800 = \text{K}3\ 200$

So his maximum profit within the constraints is K3 200.

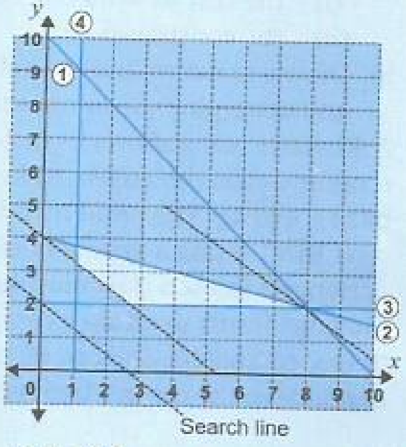

Figure 2.13

Activity 6

1 Use the graphs you drew in Activity 5 for this question.
   - a) Find the gradient of the search line for each question.
   - b) Draw the search line through each vertex.
   - c) Can you confirm the answers you calculated in Activity 5?

2 A parking lot needs to be marked into parking spaces for cars and trucks. The spaces for cars each need an area of 10 m^2 and those for trucks 30 m^2. The total area available for all the spaces is 2 000 m^2. There must be at least 20 car spaces and 20 truck spaces.
   - a) If you let the number of car spaces be x and the number of truck spaces be y, express the conditions above as inequations.
   - b) Illustrate these inequations in a diagram.
   - c) If the parking charges are K15/hour for a car and K25/hour for a truck, how many car spaces and truck spaces will give the maximum possible income?
   - d) Calculate the hourly income if $\frac{3}{4}$ of the car spaces and $\frac{1}{2}$ of the truck spaces are full.

3 A copper mine must produce the following quantities of copper: 1 000 tonnes Grade 1; 700 tonnes Grade 2; 2 000 tonnes Grade 3 and 4 500 tonnes Grade 4. Copper Levels A and B can be mined at a cost of K4 000 and K10 000 respectively per shift. The returns in tonnes per shift for each level are indicated below:

Grade	1	2	3	4
Level A	200	100	200	400
Level B	100	100	500	1 500

Let the number of shifts per week on Level A be x and Level B be y.
   - a) Determine in terms of x and y:
     - (i) the cost to work both levels per week
     - (ii) the constraints.
   - b) Draw a graph to find the values of x and y that will minimise the costs and determine the minimum cost.

4 A 48-seater plane allows its first-class passengers 60 kg luggage and economy-class passengers 20 kg. The total weight of luggage allowed is at most 1 440 kg. The profit on a first-class ticket is K300 and for economy-class K150. Using a system of equations and a graph, determine how many passengers in each class must be transported for maximum profit.

5 Mulenga wants to set up a computer centre for students. She has K75 000 to buy computers. She can set up at most 15 computers in the area she has available in the centre. Two types of computers are available: Cerebro and

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Cerebro compute
Kentek costs K7 5
a) Write a system
b) Draw a graph
c) The rates for u
K75 per sessio
optimum posi
d) What is the m
6 A furniture shop c
either woollen or
shop. There must
is K1 200 each an
three times as ma
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a) Write down th
b) Find the want
c) Use a search li
a maximum p
d) Calculate the r

Activity 6 (continued)

the more powerful and faster Kentek computer. Mulenga needs at least 5 Cerebro computers and 3 Kentek computers. Cerebro costs K6 250 and Kentek costs K7 500.

a) Write a system of inequations to represent the above information.
b) Draw a graph to determine the feasible region.
c) The rates for using the computers are K45 per session for Cerebro and K75 per session for Kentek. Draw the profit line on the graph in the optimum position.
d) What is the maximum profit per session that she can make?

6 A furniture shop owner has K42 000 to buy rugs to sell in his shop. He buys either woollen or polyester rugs. He can keep a maximum of 70 rugs in the shop. There must be at least 10 of each stock item. The price of woollen rugs is K1 200 each and the polyester rugs are K600 each. He can sell at most three times as many polyester as woollen rugs. The profit for each woollen rug is K600, and the profit for a polyester rug is K300.

a) Write down the constraints.
b) Find the wanted region by graphing the constraints.
c) Use a search line to find out how many of each rug he must purchase for a maximum profit.
d) Calculate the maximum profit.

Summary, revision and assessment

Summary

- An equation in one variable can be solved by simplifying the equation until the variable is on its own on one side of the equation. This is the solution.
- An inequation in one variable is solved in the same way, except that the direction of the inequation changes if multiplying or dividing one side of the inequation by a negative number.
- The solution to inequations in one variable can be represented on a number line.
- A linear inequation in two variables describes an area of the Cartesian plane that has a boundary line. Every point in that region is a solution of the inequation.
- Linear programming is a mathematical technique for maximising or minimising a linear function of variables such as output or cost.
- The objective function (optimisation equation) is the relationship that uses the variables to calculate a quantity that can be optimised, i.e. maximised or minimised. It is the equation connecting the variables, which leads to a solution.
- Linear programming consists of these steps:
  - Step 1: Choose variables to represent the quantities that need to be optimised.
  - Step 2: Write down the objective function.
  - Step 3: Represent all the constraints of the problem as inequations.
  - Step 4: Draw the graph showing the constraints and the feasible region.
  - Step 5: Find the gradient of the search line from the objective function.
  - Step 6: Draw the search line through each vertex of the feasible region to find the vertex that will optimise the result.
  - Step 7: Substitute the coordinates of the vertex into the objective function to find either the maximum or minimum values, or to find the solution.

Revision exercises (Remedial)

1 The perimeter of a square must be less than 160 m. What is the maximum length of a side in metres? Write down an inequation to show this.

2 Write down the equation that forms each boundary line of the feasible region shown in Fig. 2.14.

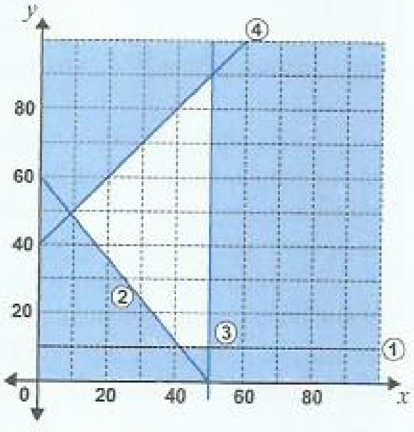

Figure 2.14

Revision exer

1 Write down the
a) $2x - 33 > 11$
d) $\frac{2x-3}{2} + \frac{4-3}{5}$
2 The number of
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the solution.

Summary, revision and assessment (continued)

Revision exercises

1 Write down the solution and represent it on a number line.

a) $2x - 33 > 11$ b) $\frac{x-3}{2} < -5$ c) $-2 < \frac{6-2x}{3} < 4$

d) $\frac{2x-3}{2} + \frac{4-3x}{5} \geq \frac{x+2}{4}$ e) $4\frac{1}{2} < \frac{3x+2}{4} - \frac{x+2}{2}$

2 The number of small tables at a wedding venue is represented by x and the number of large tables is represented by y. Each small table seats 6 people and each large table seats 10 people.

a) Write down an inequation to show the maximum number of people n that can be seated at the restaurant.

b) If there are at most 120 people attending a wedding, write down an inequation to show the possible values of x and y.

c) Write this inequation with y as the subject.

d) Draw the graph and indicate the feasible region to show the possible values of x and y.

3 There are 50 learners on a bus. If 6 more boys get on the bus the number of boys will be three times that of the girls. How many girls were on the bus to start off with?

4 A vehicle manufacturer produces two types of motorcycles, the Speedster and the Forcestar. These are sold at a profit of K20 000 per Speedster and K10 000 per Forcestar. The Speedster requires 150 hours for assembly, 50 hours for finishing and 10 hours for checking and testing. The Forcestar requires 60 hours for assembly, 40 hours for finishing and 20 hours for checking and testing. The total number of hours per month is: 30 000 hours for the assembly department, 13 000 hours for the finishing department and 5 000 hours for the checking and testing department.

Let x be the number of Speedster and y be the number of Forcestar motorcycles they manufacture each month.

a) Write down the constraints.

b) Draw a graph to represent the constraint inequations and indicate the feasible region.

c) Write down the objective function in terms of x and y. Find the gradient of the search line.

d) How many motorcycles of each model must be produced in order to maximise the monthly profit?

e) What is the maximum monthly profit?

Summary, revision and assessment (continued)

Assessment

1 Miyoba has K300 to spend on some clothes. He needs a pair of jeans which cost K120 and he can buy shirts for K25 each. How many shirts can he buy? (4)

2 The velocity v in metres per second (m/s) of a ball thrown straight up in the air is given by the equation $v = 50 - 5t$, where t is the time in seconds.

- a) At what times will the velocity be less than 30 m/s? (4)
- b) At what times will the velocity be between 5 m/s and 15 m/s? (4)
- c) Can the velocity be greater than 80 m/s? Explain. (2)

3 Use simultaneous equations to solve this problem: The sum of Wamunyima's age and Penjani's age is 60. Six years ago, Wamunyima was three times as old as Penjani. Find both of their ages now. (6)

4 In a certain week an electronic tablet manufacturer makes two types of tablets: gold and black. At most 60 of the gold tablets and 50 of the black tablets can be manufactured in a week. At least 80 tablets must be produced in a week to cover the costs. It takes $\frac{2}{3}$ hour to assemble a gold tablet and $\frac{1}{2}$ hour to assemble a black tablet. The factory works a maximum of 60 hours per week.

- a) Allocate the variables and write down the constraint inequations. (6)
- b) If the profit on a gold tablet is K400 and on a black tablet is K500, write down the equation that represents the amount of profit that they can earn. (2)
- c) Draw a graph and plot the constraints. Indicate the feasible region. (4)
- d) Draw a search line on the graph that represents the objective function (optimisation equation). (2)
- e) Use the graph to determine how many gold tablets and how many black tablets are to be manufactured for maximum profit. (2)
- f) What is the maximum weekly profit? (2)

[18]

TOPIC

3

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Velocity–time grap

Starter activ

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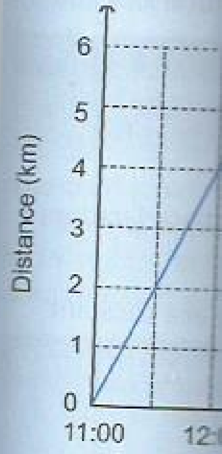

Figure 3.1

1 Discuss with y
- a) Why do we
- b) The graph s shows displ scalar?

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3 a) During whi
- b) What is her

4 For how long d

5 a) What is her
- b) What is her

6 Why is the slop

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[18]

TOPIC 3

Travel graphs

Sub-topic	Specific Outcomes
Velocity–time graphs (curves)	• Calculate the displacement in a velocity–time graph.

Starter activity

Chileshe goes for a walk from her house. She sketches the graph in Fig. 3.1 to show how far she is from her starting point at different times.

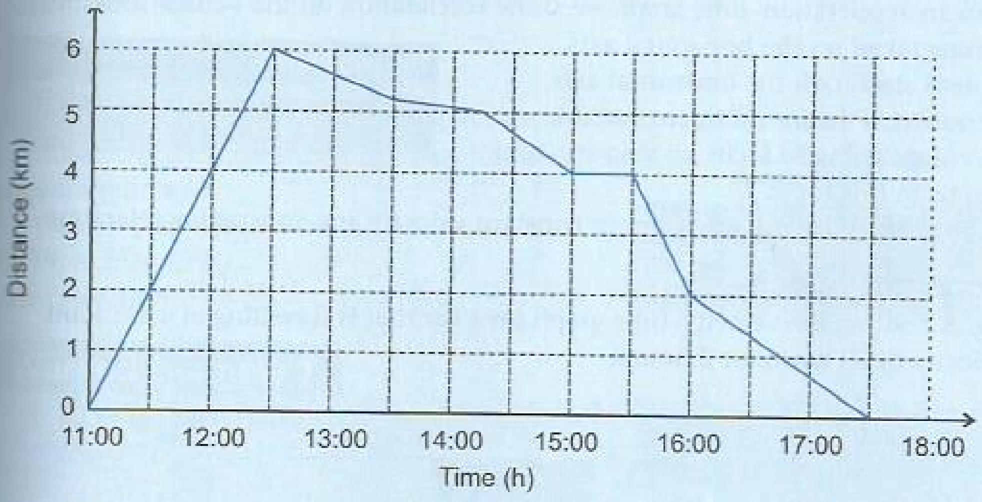

Figure 3.1

1 Discuss with your partner.
  a) Why do we show time on the horizontal axis on this kind of graph?
  b) The graph starts at 0 km and ends at 0 km. Do you think that the graph shows displacement, which is a vector quantity, or distance, which is scalar?
2 What is her furthest distance from her home during the walk?
3 a) During which time period does she walk the fastest?
  b) What is her average speed during this time period?
4 For how long does she stop during the walk? Explain how you know this.
5 a) What is her average speed for the whole walk?
  b) What is her average velocity for the whole walk?
6 Why is the slope decreasing in the last part of the graph?

SUB-TOPIC 1

Velocity–time graphs

Revision of travel graphs

Travel graphs are line graphs that are used to describe the motion of objects such as buses, cars, trains, cyclists and people walking. A travel graph shows a journey or a trip.

We can depict travel graphs in various ways depending on which quantities we want to show on the graph.

- In a velocity–time graph, we represent the velocity on the vertical axis and the time taken on the horizontal axis.
- In a distance–time graph, we represent the distance travelled (displacement) on the vertical axis and the time taken on the horizontal axis.
- In an acceleration–time graph we show acceleration on the vertical axis and time taken on the horizontal axis.

Time is always on the horizontal axis, because it is the independent variable.

We are going to focus on velocity–time graphs in this topic.

First we revise the situations of constant velocity and constant acceleration.

Did you know?

In Physics, the study of the motion of objects is called kinematics.

Constant velocity

Fig. 3.2 shows the velocity–time graph for a car that is travelling at a constant velocity of 70 km/h for 2 hours.

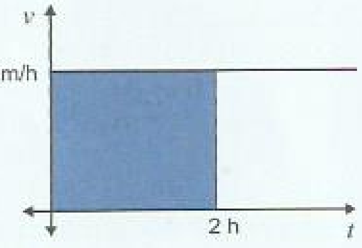

Figure 3.2

The area under this straight-line graph is rectangular in shape.

The shaded area = 2 h × 70 km/h = 140 km.

The area under a velocity–time graph gives us the displacement of the car.

As the velocity is constant, there is no acceleration. The gradient of the graph is 0.

Note

The area under a velocity–time graph indicates displacement.

Constant accele

Now consider the ca
accelerates steadily fr
of 0 to 70 km/h in 1
Figure 3.3 shows this

Again, the area un
give the displacemen
velocity is in kilomet
time is in seconds. W
the units to find the

Convert 70 km/h to
Area = $\frac{1}{2}$base × heigh
Area = $\frac{1}{2}$(10) ×19.44
Displacement = 97.2

The slope of a veloci
because velocity is m
measured in m/s².

Here the car has
to 19.44 m/s in 10 s
∴ acceleration = 19.

Worked examp

1 Look at the vel
road.

a) Describe th
b) Calculate t
c) Calculate t
d) What is th

motion of objects such
graph shows a journey

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dient of the graph is 0.

Constant acceleration

Now consider the case when a car accelerates steadily from a velocity of 0 to 70 km/h in 10 seconds. Figure 3.3 shows this situation.

Again, the area under the graph will give the displacement. Notice that the velocity is in kilometres per hour and time is in seconds. We need to convert the units to find the displacement.

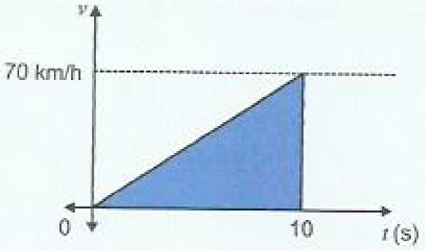

Figure 3.3

Convert 70 km/h to m/s: $(70 \times 1\ 000) \div 60 \div 60 = 19{,}44$ m/s

Area = $\frac{1}{2}$base × height

Area = $\frac{1}{2}(10) \times 19.44 = 97.2$

Displacement = 97.2 m

The slope of a velocity–time graph gives the acceleration. This makes sense because velocity is measured in m/s while acceleration is $\frac{\text{change in velocity}}{\text{change in time}}$ and is measured in m/s^2.

Here the car has accelerated from zero to 19.44 m/s in 10 s

$\therefore$ acceleration = $\frac{19.44 - 0}{10} = 1.94$ m/s^2

Note

The slope of a velocity–time graph indicates acceleration.

Worked example 1

1 Look at the velocity–time graph (Fig. 3.4) for a motorbike driving along a road.

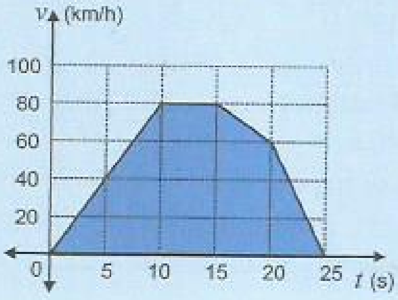

Figure 3.4

a) Describe the movement of the motorbike over the whole time period.
b) Calculate the acceleration from $t = 0$ s to $t = 10$ s.
c) Calculate the acceleration between $t = 15$ s and $t = 20$ s.
d) What is the total displacement of the motorbike during the 25 seconds?

Worked example 1 (continued)

Answers

a) The motorbike accelerates steadily from a stationary position to 80 km/h in 10 s and then maintains a constant velocity for 5 s before decreasing the velocity to 60 km/h for 5 s, and then decelerating to a velocity of zero for the last 5 s.

b) The acceleration is the slope (gradient) of the graph.
First convert km/h to m/s: 80 km/h = 80 000 ÷ 3 600 = 22.22 m/s
Acceleration = 22.22 m/s ÷ 10 s = 2.22 m/s2

c) At $t = 20$ s, the velocity is 60 km/h = 60 000 ÷ 3 600 = 16.67 m/s
Acceleration = $\frac{16.67 - 22.22}{5} = \frac{-5.55}{5} = -1.11$ m/s^2
Notice that the acceleration is negative here. This shows that the motorbike is slowing down.

d) The displacement is the area under the graph, which can be broken into parts, as shown in Fig. 3.5. The four shapes are two triangles, a rectangle and a trapezium.

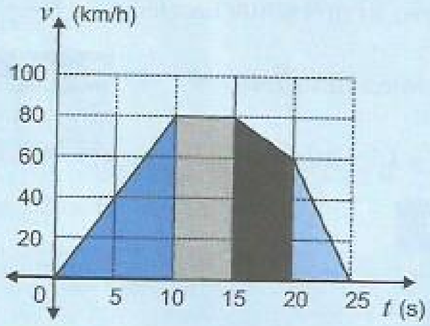

Figure 3.5

From $t = 0$ to $t = 10$: displacement = $\frac{1}{2}(10) \times 22.22$ m/s = 111.1 m
From $t = 10$ to $t = 15$: displacement = 22.22 m/s × 5 s = 111.1 m
From $t = 15$ s to $t = 20$ s:
displacement = area of trapezium = $\left(\frac{22.22 + 16.67}{2}\right)5 = 97.23$ m
From $t = 20$ s to $t = 25$ s, displacement = $\frac{1}{2}(5) \times 16.67 = 41.68$ m
Sum of the four areas = 361.11 m

Activity 1

1 In each case sketch a velocity–time graph and find the displacement in metres or kilometres.
  a) A car travels at a steady velocity of 85 km/h for 3 hours.
  b) A motorbike decelerates (slows down) steadily from a velocity of 64 km/h until it stops 5 s later.

Activity 1 (co

c) A bicycle a
d) A car dece

2 An athlete do
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in km/h and t
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  training ru
  b) Compare t
  in the first
  run to the
  the time be
  and 45 min
  c) Calculate th
  that she co
  d) What is her
  in km/h?

3 Fig. 3.7 shows t
and time for a c
attempt to reac
of 20 m/s. Use t
to answer the q
  a) Convert 20 r
  b) Explain whe
  cyclist was a
  sustain a vel
  of 20 m/s.
  c) Calculate the
  distance that
  cyclist covere
  d) Calculate the

Velocity as a

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function of time, or w

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Activity 1 (continued)

c) A bicycle accelerates steadily from 5 km/h to 30 km/h over a period of 30 s.

d) A car decelerates steadily from 90 km/h to 50 km/h over a period of 9 s.

2 An athlete does a training run where she needs to increase her speed throughout the run. She uses a GPS watch to measure her velocity and the time of a training run. She draws the graph in Fig. 3.6 to summarise the run. In the figure, velocity v is in km/h and time t is in min.

a) Describe the athlete's training run.

b) Compare the acceleration in the first 15 min of the run to the acceleration in the time between $t = 35$ min and 45 min.

c) Calculate the total distance that she covers during the run.

d) What is her average speed in km/h?

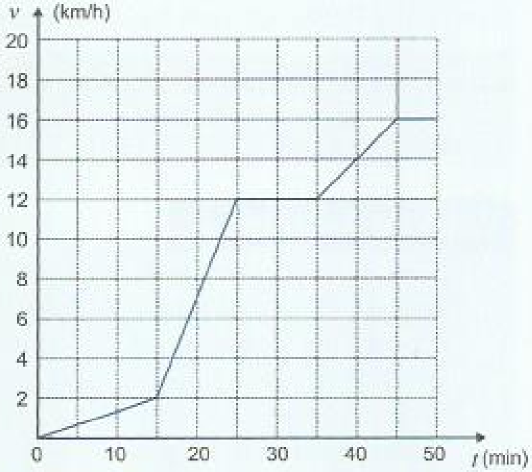

Figure 3.6

3 Fig. 3.7 shows the velocity and time for a cyclist's attempt to reach a velocity of 20 m/s. Use the graph to answer the questions.

a) Convert 20 m/s to km/h.

b) Explain whether the cyclist was able to sustain a velocity of 20 m/s.

c) Calculate the total distance that the cyclist covered in this time period.

d) Calculate the acceleration between $60 \leq t \leq 80$.

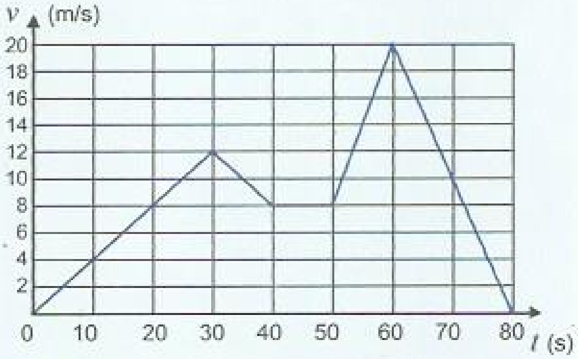

Figure 3.7

Velocity as a function of time

We can describe velocity–time graphs where acceleration is constant by a formula for a straight line function:

$$v(t) = v_0 + a\Delta t$$

where $v(t)$ is the velocity or final velocity, v_0 is the initial velocity, a is the acceleration, Δt is the change in time. This is the equation for velocity as a function of time, or $v(t)$.

Compare this function $v(t) = a\Delta t + v_0$ to the standard form of the linear function $y = mx + c$.

- The intercept with the vertical axis c will be the starting velocity v_0.
- The gradient of the graph m is equal to a, the acceleration, which is constant for linear functions.

You do not need to use this formula each time. You can simply use the standard form of the linear function for all velocity–time functions.

Note

The units must be consistent. If t is measured in seconds, then v must be distance covered per second.

Worked example 2

1 A cyclist accelerates from rest to a velocity of 750 m/min during the time period $t = 0$ to $t = 15$ min. He maintains this velocity for 10 min, before decelerating steadily at -375 m/min^2 over 5 min.

a) Draw a sketch graph to show the cyclist's motion.

b) Write three equations in the form of a linear function for the three segments of the trip.

c) Calculate the total displacement.

2 A ball rolls along a straight line at 15 m/s for $0 \le t < 2$ and the velocity increases to 25 m/s steadily during the period 25 m/s for $2 \le t \le 5$ where t is measured in seconds.

a) Sketch the graph of the velocity function.

b) Find the displacement of the ball for the time period $0 \le t \le 5$.

c) Write down two different equations for $v(t)$ where $0 \le t < 2$ and $0 \le t \le 5$.

Answers

1 a)

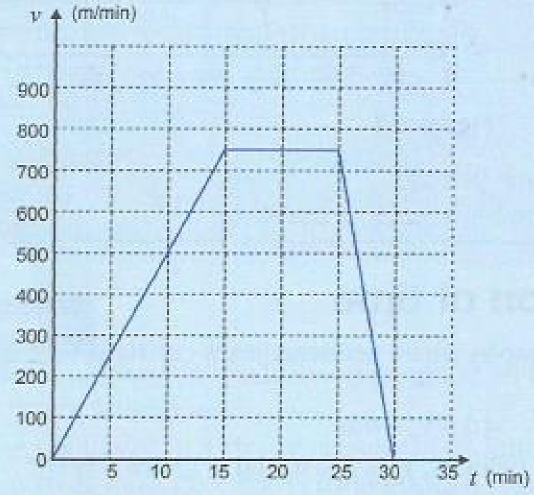

Figure 3.8

Worked exan

b) First segm

$v_0 = 0$; $v(t)$

$a = \frac{\Delta v}{\Delta t} =$

$v(t) = 50\Delta t$

Second seg

$v_0 = 750$ m

Equation:

Third segm

$v_0 = 750$ m

Equation:

c) Total displa

2 a) Graph:

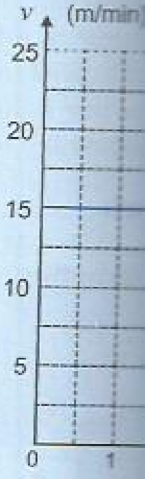

Figure 3.9

b) Total displac

c) Where $0 \le t$

Where $2 \le t$

Calculate $a =$

$v(t) =$

Worked example 2 (continued)

b) First segment:

$v_0 = 0$; $v(t) = 750$ m/min; $\Delta t = 15$ min

$a = \frac{\Delta v}{\Delta t} = \frac{750}{15} = 50$ m/s²

$v(t) = 50\Delta t$

Second segment:

$v_0 = 750$ m/min; $v(t) = 750$ m/min; $\Delta t = 10$ min

Equation: $v(t) = 750$

Third segment:

$v_0 = 750$ m/min; $a = -375$ m/min²; $\Delta t = 5$ min

Equation: $v(t) = 750 - 375\Delta t$

c) Total displacement $= [\frac{1}{2}(15)(750] + (10 \times 750) + [\frac{1}{2}(5)(750)]$

$= 15\ 000$ m $= 15$ km

2 a) Graph:

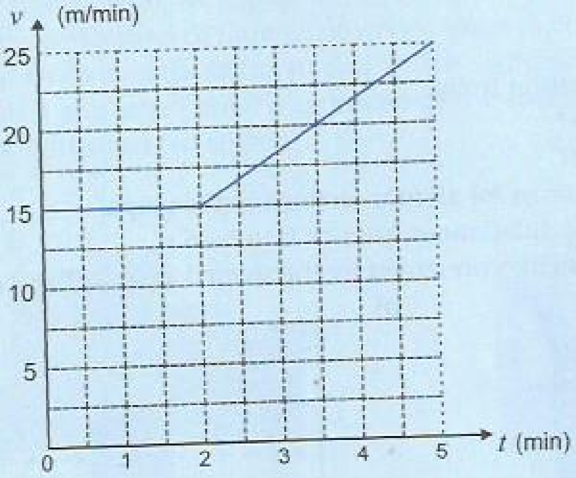

Figure 3.9

b) Total displacement: $(15 \times 2) + \frac{1}{2}(3)(15 + 25) = 90$ m

c) Where $0 \leq t < 2$: $a = 0$ so the equation is $v(t) = 15$

Where $2 \leq t \leq 5$:

Calculate $a = \frac{\Delta v}{\Delta t} = \frac{25 - 15}{5 - 2} = \frac{10}{3}$

$v(t) = 15 + \frac{10}{3}\Delta t$

Activity 2

1 The velocity of a car as it goes up a hill is given by the equation $v = 20 - 3t$, where t is the time in seconds and v is measured in metres per seconds.

a) Draw a sketch graph of v against t from $t = 0$ to $t = 10$ s.

b) What is the displacement during this time?

2 Fig. 3.10 shows the velocity function for a moving object.

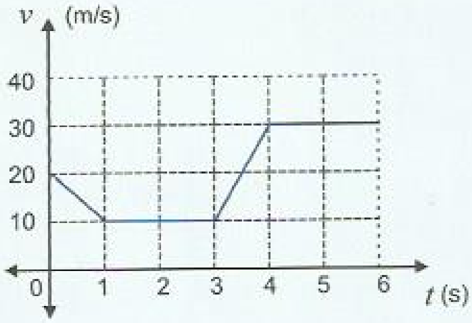

Figure 3.10

a) What is the acceleration from:

(i) $t = 0$ s to $t = 1$ s?

(ii) $t = 3$ s to $t = 4$ s?

b) Write velocity functions for all four sections of the graph.

c) Calculate the total displacement from $t = 0$ to $t = 6$ s.

3 Calculate the displacement represented by the shaded areas in these graphs:

a)

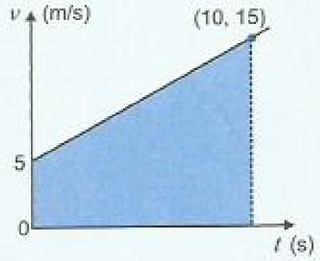

Figure 3.11

b)

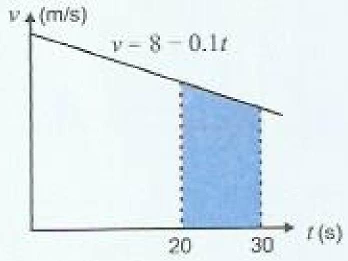

Figure 3.12

c)

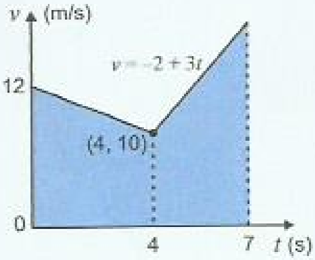

Figure 3.13

d)

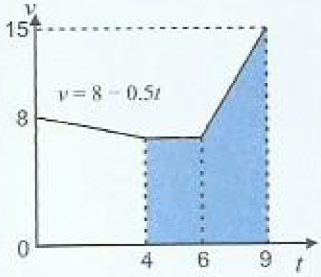

Figure 3.14

Activity 2 (c

4 An athlete p

- Run at 10
- Accelerat
- Keep a co
- Accelerat
- Keep a co
- Decelerat

a) Sketch a

b) Calculat

c) What is
12 km/h

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In Topic 1 yo
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Figure 3.15

Left sums:

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Activity 2 (continued)

4 An athlete plans to do the following training run:
- Run at 10 km/h for 10 min
- Accelerate steadily over 30 s to 12 km/h
- Keep a constant velocity for 10 min
- Accelerate steadily to 14 km/h over 30 s
- Keep a constant velocity for 10 min
- Decelerate steadily for 1 min to come to a stop

a) Sketch a graph of this situation.

b) Calculate the total displacement that he would cover during this run.

c) What is the velocity function $v(t)$ for the period of acceleration from 12 km/h to 14 km/h?

Acceleration which changes constantly

We know how to work with graphs where there is a constant acceleration. But in reality, the acceleration of moving objects is likely to change constantly. We need a different kind of mathematics to deal with a constantly changing situation.

In Topic 1 you learnt how to approximate the area under a curve, using left-sided and right-sided rectangles.

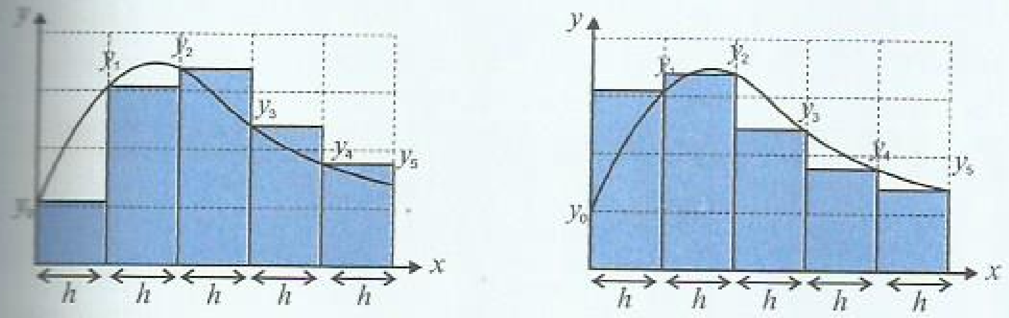

Figure 3.15

Left sums:
Area under curve = $h(y_0 + y_1 + y_2 + y_3 + y_4)$

Right sums:
Area under curve = $h(y_1 + y_2 + y_3 + y_4 + y_5)$

Check these formulae carefully against the graphs in Fig. 3.15. Make sure that you understand why each formula uses the variables it does.

We apply a similar method to velocity–time functions.

Worked example 3

The table and graph in Fig. 3.16 show the velocity of a car as it goes from one traffic light to the next.

t (s)	0	2	4	6	8	10	12
v (m/s)	0	5	8	9	8	5	0

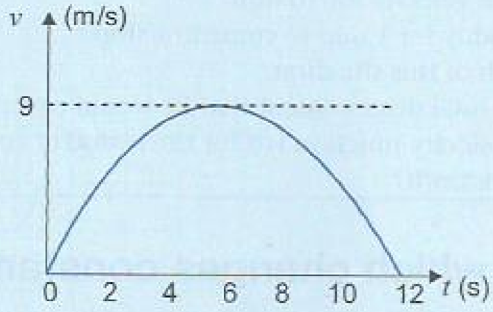

Figure 3.16

$t \in [0, 12]$

Estimate the displacement of the car in metres by dividing the area under the graph into six sub-intervals of 2 seconds. Use both left sums and right sums of rectangles.

Answer

The sub-intervals of 2 seconds are shown in Fig. 3.17.

$\Delta t = 2$ s

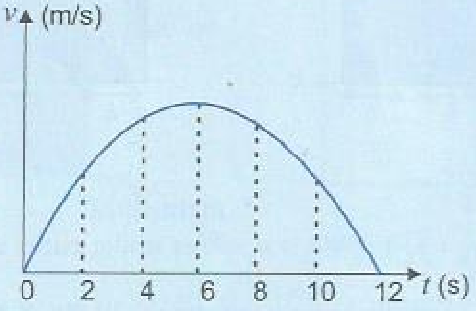

Figure 3.17

We then draw rectangles under the curve with the left or right top vertex of each rectangle touching the curve. The table gives us the values of the function $v(t)$ at each interval, so we don't need to find the equation $v(t)$.

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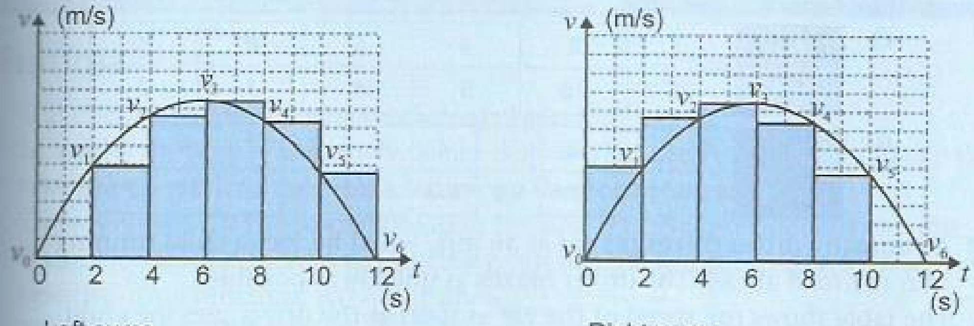
Worked example 3 (continued)



Left sums

Right sums

Figure 3.18

From the table:

t (s)	0	2	4	6	8	10	12
v (m/s)	0	5	8	9	8	5	0

Left sums:

$$\text{Area} = \Delta t(v_0 + v_1 + v_2 + v_3 + v_4 + v_5)$$
$$= 2(0 + 5 + 8 + 9 + 8 + 5)$$
$$= 70$$

Displacement = 70 m

Right sums:

$$\text{Area} = \Delta t(v_1 + v_2 + v_3 + v_4 + v_5 + v_6)$$
$$= 2(5 + 8 + 9 + 8 + 5 + 0)$$
$$= 70$$

Displacement = 70 m

These sums are identical, because the graph is symmetrical.

Note

- We could find more accurate estimates by using more values and narrower strips.
- The width of the strip if $t \in [a, b]$ is $\Delta t = \frac{b-a}{n}$, where n is the number of strips or sub-intervals.

Activity 3

1 The tables of values below each show the $v(t)$ function and the values of the function at certain sub-intervals of time. For each of the following:

(i) Draw a velocity–time graph on grid paper.

(ii) Describe what happens during the given time interval.

(iii) Estimate the distance travelled using the rectangle method.

a)

t (s)	0	1	2	3	4	5	6	7	8
v (m/s)	0	5.0	7.1	8.7	10.0	11.2	12.2	13.2	14.1

b)

t (s)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
v (m/s)	5.0	6.9	8.6	10.1	11.4	12.5	13.4	14.1	14.6	14.9	15.0

Activity 3 (continued)

c)

t (s)	0	2	4	5	8	10
v (m/s)	25	16	9	4	1	0

d)

t (s)	0	1	2	3	4	5	6	7	8	9
v (m/s)	20.0	14.0	11.5	9.6	8.0	6.6	5.3	4.1	3.0	2.0

2 A company driver drives his car at 36 m/s, when he sees a child running into the road ahead. The driver brakes as quickly as possible.
The table shows the speed of the car as soon as the driver sees the child.

t (s)	0	1	2	3	4	5	6	7
v (m/s)	36	36	34.8	29.9	23.2	15.2	4.8	0

a) Draw a velocity–time graph on grid paper to show this situation.
b) Calculate the displacement during braking using sub-intervals of 7 seconds.
c) Draw up another table showing sub-intervals of 0.5 s. Calculate the displacement using these sub-intervals.

3 The velocity of an object is given by the function $v = t^2$, over the interval $0 \leq t \leq 8$.
a) Draw a sketch graph of the situation.
b) Describe the motion of the object over this time period.
c) Calculate the displacement of the object using
  (i) 8 sub-intervals
  (ii) 16 sub-intervals
d) The actual area under this curve in this interval is equal to $\frac{1}{3}t^3 = 24$ units. Comment on your answers in c).

4 Consider the area under a graph of $v(t) = -t^2 + 5$, in the interval $0 \leq t \leq 2$.
a) Draw a sketch graph of the situation.
b) Calculate the displacement of the object using
  (i) 5 sub-intervals
  (ii) 10 sub-intervals
c) The actual displacement of the object during this time period is equal to $-\frac{1}{3}t^3 + 5t$. Comment on your answers in b).

Summary

- Travel graphs re
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- Time is always c
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TOPIC 3

Summary, revision and assessment

Summary

- Travel graphs represent the motion of objects.
- In a velocity–time graph, the velocity is shown on the vertical axis and the time taken to travel that distance is shown on the horizontal axis.
- Time is always on the horizontal axis, because it is the independent variable.
- If the acceleration is constant, then we can use the linear formula to describe a velocity–time function: $v(t) = v_0 + a\Delta t$
- The slope of a $v(t)$ function represents the acceleration and the area under the graph represents the displacement.
- If the acceleration is constantly changing the $v(t)$ function will be represented by a curved graph rather than straight-line segments. The $v(t)$ function is then a more complex function.
- The displacement under a curve is represented by the area under a curve.
- The area under a curve for a $v(t)$ function with changing acceleration cannot be calculated as accurately as with a linear $v(t)$ function.
- We can estimate the area between two intervals by dividing it into sub-intervals and calculating the area of rectangles or trapezium shapes under the curve. The more sub-intervals there are, the closer the estimate will become to the actual area.

Summary, revision and assessment (continued)

Revision exercises

1 A runner starts at the bottom of the hill at a sprint and then runs up the hill with a constant acceleration. The velocity is given by the equation $v = 5 - 0.1t$, where t is the time in seconds and v is measured in metres per second.

a) Draw a sketch graph of v against t from $t = 0$ to $t = 30$ s.

b) What is the displacement during this time?

2 A cyclist accelerates steadily from rest to a velocity of 200 m/min during the time period t_0 to $t = 8$ min. She then maintains a constant velocity for 15 min. Finally she comes to a stop by steadily decelerating for 2 min.

a) Draw a sketch graph to show the cyclist's motion.

b) Write three equations in the form of a linear function for the three segments of the journey.

c) Calculate the total displacement.

3 Look at Fig. 3.19.

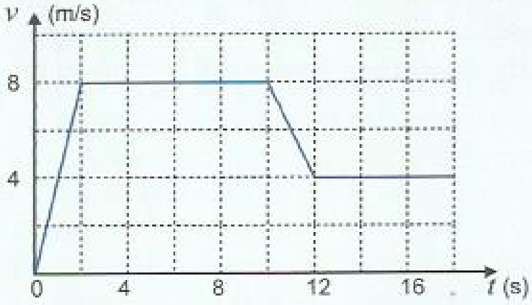

Figure 3.19

a) Write four equations in the form $v(t) = v_0 + a\Delta t$ for the four segments of the trip.

b) Calculate the total displacement.

4 Calculate the displacement in each of these functions over the time periods given. Use $n = 8$ sub-intervals in each case.

a) $v(t) = 4 - t^2;\ 2 \le t \le 6$

b) $v(t) = x^2 + 4;\ t \in [1, 2]$

c) $v(t) = -t^2 + 3;\ t \in [1, 4]$

d) $v(t) = -t^3;\ 1 \le t \le 3$

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5 a) Explain ho
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5 **a)** Explain how the sum of rectangle approximations of the area of a region under a curve changes as the number of sub-intervals increases.

b) The interval [1, 4] on a graph is our area of interest. Suppose we divide it into 10 sub-intervals.

(i) What is the sub-interval length?

(ii) List the x-values at the boundaries of the sub-intervals.

Summary, revision and assessment (continued)

Assessment

1 A water reservoir supplies water to a nearby town at a flow rate in cubic metres per hour as shown in Fig. 3.21.

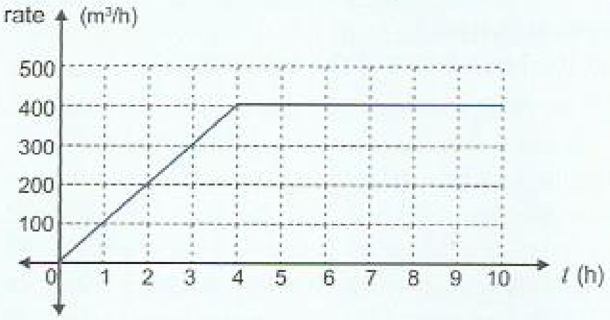

Figure 3.20

a) Calculate the amount of water that flows out in the time 0 to 4 h.
b) Calculate the amount of water that flows out in the interval [8, 10].
c) Does more water flow out of the reservoir in [0, 4] or [4, 6]?

2 Approximate the area of the region under the graph of $v(t) = 100 - t^2$ over the interval [0, 10] and with $n = 10$ sub-intervals. Use the midpoint of each sub-interval.

3 Complete the steps for the given function, interval and the value of n (number of sub-intervals):

(i) Sketch the $v(t)$ graph.
(ii) Calculate Δt and the values t_0; t_1; t_2; .... t_n
(iii) Draw up a table of values for the values of $v(t)$ at t_0; t_1; t_2; .... t_n
(iv) Calculate the left and right sums.
(v) Determine which sum overestimates and which sum underestimates the actual displacement.

a) $v(t) = t^2 - 1$; $t \in [2, 4]$; $n = 4$
b) $v(t) = 2t^2$; $t \in [1, 6]$; $n = 10$

The actual formulas for displacement are as follows:

a) $s = \frac{1}{3}t^3 - t$
b) $s = \frac{2}{3}t^3$

TOPIC

4

Sub-

Introduction to ve

Addition and subt

Translations

Scalar multiplicati

Collinearity

Vector geometry

Starter activ

Look at the map

Figure 4.1

a) Describe two
b) Which of the

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4, 6]?
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TOPIC 4

Vectors in two dimensions

Sub-topic	Specific Outcomes
Introduction to vectors	- Describe a vector. - Represent and denote a vector.
Addition and subtraction	- Add and subtract vectors.
Translations	- Apply translations on vectors and find magnitude.
Scalar multiplication	- Multiply vectors by scalars.
Collinearity	- Determine collinearity of points.
Vector geometry	- Solve geometrical problems involving vectors.

Starter activity

Look at the map in Fig. 4.1

Figure 4.1

a) Describe two routes you could use to travel from Mansa to Mpika.
b) Which of the two routes is shorter?

SUB-TOPIC 1 Introduction to vectors

Describe a vector

You learnt in previous grades that a vector is a quantity that has both magnitude (size) and direction.

- Quantities such as displacement, translation, force, acceleration and velocity are examples of vectors since they have both magnitude and direction.
- Quantities such as time, energy and mass are examples of scalar quantities as they only have magnitude, not direction.

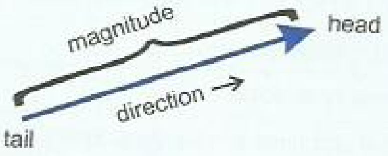

Figure 4.2

New word

translation: change in position

Some examples:

Vector quantity (Magnitude and direction)	Scalar quantity (Magnitude only)
A movement of 5 m to the right	A movement of 5 m
An object with a weight of 100 N (Direction downwards to centre of the Earth)	An object with a mass of 10 kg
Increase or decrease in temperature	Temperature
Velocity	Speed
Displacement	Distance

Represent a vector

Because a vector quantity always has magnitude and direction it can be represented in a drawing as a directed line segment with a certain length.

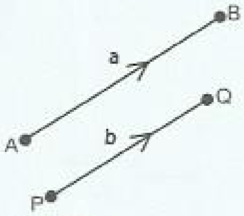

Figure 4.3

New words

tail: initial (starting) point of a vector
head: terminal (end) point of a vector

$\overrightarrow{AB}$ is a directed line segment of a certain length, with initial point A (tail) and terminal point B (head).

- The length of
- The directio

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the vector **b**,

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1. Don't conf
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Equal vec

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$\overrightarrow{AB} = \begin{pmatrix}5\\3\end{pmatrix}$ and $\overrightarrow{BA}$

Also, $\overrightarrow{BA} = -\overrightarrow{AB} =$

Similarly, $\overrightarrow{CD} = -$

- The length of the line segment indicates the magnitude of the vector.
- The direction of the arrowhead indicates the direction of the vector.

We often label a directed line segment by the vector it represents. In Fig. 4.3, the line segment $\overrightarrow{AB}$ represents the vector **a**. Similarly, the line segment $\overrightarrow{PQ}$ represents the vector **b**, or if writing by hand, $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

Note that vector $\overrightarrow{BA}$ is not the same as $\overrightarrow{AB}$.

Vectors can be represented in the form $\begin{pmatrix}x\\y\end{pmatrix}$, where the top number, x, represents horizontal movement, and the bottom number, y, represents vertical movement In Fig. 4.4, the vector $\mathbf{a} = \begin{pmatrix}3\\5\end{pmatrix}$ has a horizontal component of +3 and a vertical component of +5.
The arrowhead is important, as it shows the direction.

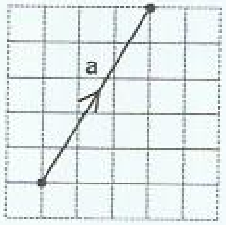

Figure 4.4

Note

1. Don't confuse vectors written in coordinate form $\begin{pmatrix}x\\y\end{pmatrix}$ with fractions.
2. Don't confuse vector coordinates $\begin{pmatrix}x\\y\end{pmatrix}$ with the calculation for the gradient of a straight line.

Equal vectors

Two vectors are equal if they have the same magnitude and direction, no matter what their initial point is. The two vectors in Fig. 4.5 are equal, so **a** = **b**.

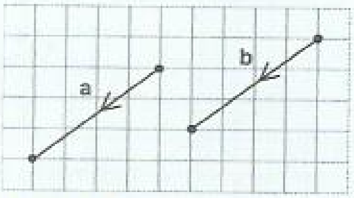

Figure 4.5

The direction of vectors

The direction of vectors is very important. In Fig. 4.6, $\overrightarrow{BA}$ is opposite in direction to $\overrightarrow{AB}$, even though they are the same size and parallel to each other.

$\overrightarrow{AB} = \begin{pmatrix}5\\3\end{pmatrix}$ and $\overrightarrow{BA} = \begin{pmatrix}-5\\-3\end{pmatrix}$

Also, $\overrightarrow{BA} = -\overrightarrow{AB} = -\begin{pmatrix}5\\3\end{pmatrix} = \begin{pmatrix}-5\\-3\end{pmatrix}$

Similarly, $\overrightarrow{CD} = -\overrightarrow{DC} = -\begin{pmatrix}-5\\3\end{pmatrix}$
$= \begin{pmatrix}5\\-3\end{pmatrix}$

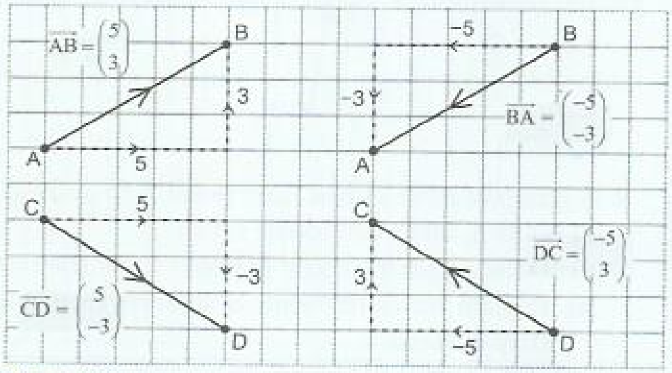

Figure 4.6

Worked example 1

Write down the vectors represented in Fig. 4.7 in coordinate form.

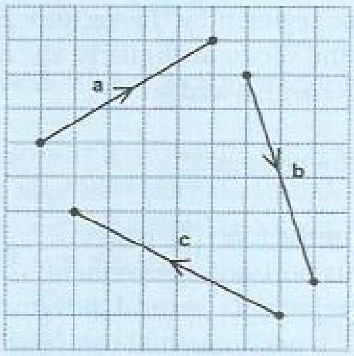

Figure 4.7

Answer

$\mathbf{a} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$; $\mathbf{b} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}$; $\mathbf{c} = \begin{pmatrix} -6 \\ 3 \end{pmatrix}$

Activity 1

1 Write each vector in Fig. 4.8 in coordinate form.

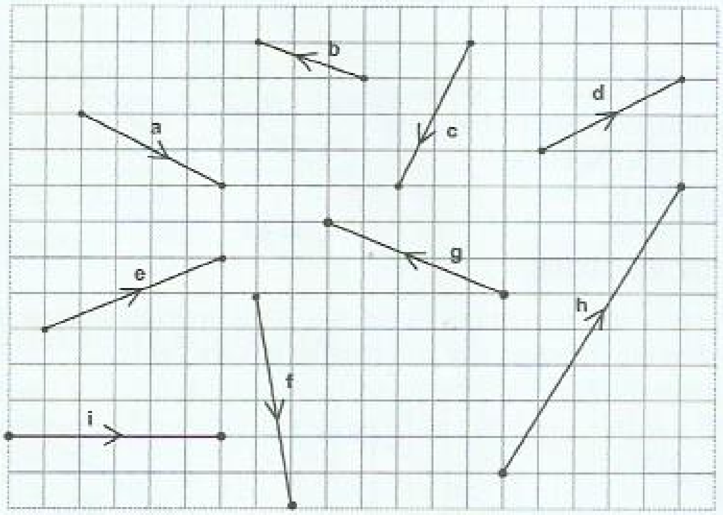

Figure 4.8

2 A group of athletes run from point A to point B. Point B is 3 km to the east of point A, and 5 km to the south.

a) Draw a vector on grid paper to show their displacement.

b) Explain how this vector is different from a diagram showing the distance they covered.

form.

3 km to the east

nt.
owing the distance

SUB-TOPIC 2 Addition and subtraction

Add vectors

Adding vectors using diagrams

The sum of two or more vectors is called the resultant of the vectors. We add vectors by using the triangle method or the parallelogram method.

New word

resultant (of vectors): the sum of two or more vectors

How to add two vectors, a and b, using the triangle rule

The sum of two separate vectors, **a** and **b**, is shown in Fig. 4.9.

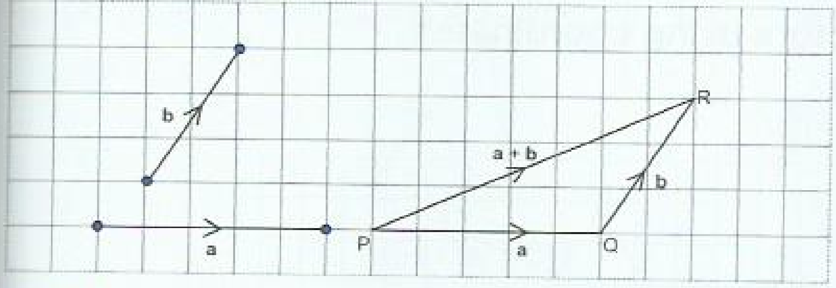

Figure 4.9

Step 1: Draw $\overrightarrow{PQ}$ equal to **a**
Step 2: Draw $\overrightarrow{QR}$ equal to **b**, in a head-to-tail arrangement with $\overrightarrow{PQ}$.
Step 3: Join points P and R. $\overrightarrow{PR}$ is a vector forming the third side of ΔPQR.

$$\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR} = \mathbf{a} + \mathbf{b}$$

This is known as the **triangle rule** for the addition of vectors.

Note

The direction of $\overrightarrow{PR}$ is the same as the direction of $\overrightarrow{PQ}$ followed by $\overrightarrow{QR}$. That is, if you moved from P to Q and then from Q to R, your overall change in position would be from P to R. The plus symbol for addition, +, in vectors means "followed by".

How to add two vectors, a and b, which start at the same point, using the parallelogram rule

Fig. 4.10 shows a situation in which vectors **a** and **b** both start at the same point.

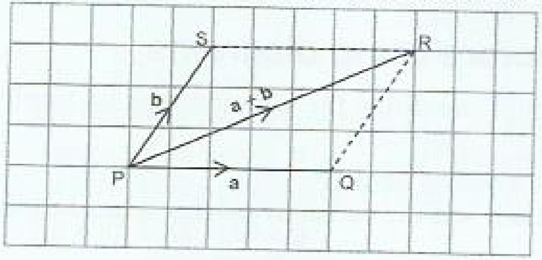

Figure 4.10

Step 1: Draw $\overrightarrow{SR}$ equal and parallel to $\overrightarrow{PQ}$.
Step 2: Draw $\overrightarrow{QR}$ equal and parallel to $\overrightarrow{PS}$. This creates a parallelogram.
Step 3: Join points P and R. $\overrightarrow{PR}$ is a vector

$$\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{PS} = \mathbf{a} + \mathbf{b}$$

This is known as the **parallelogram rule** for the addition of vectors.

The opposite sides of a parallelogram are parallel and equal vectors

$\therefore \overrightarrow{PQ} = \overrightarrow{SR} = \mathbf{a}$ and $\overrightarrow{PS} = \overrightarrow{QR} = \mathbf{b}$

If we apply the triangle rule we get:

$\overrightarrow{PR} = \overrightarrow{PS} + \overrightarrow{SR}$ or $\overrightarrow{PQ} + \overrightarrow{QR}$

$= \mathbf{b} + \mathbf{a} \quad = \mathbf{a} + \mathbf{b}$

Adding vectors using coordinates

Look at Fig. 4.11.
$\overrightarrow{PQ} = \begin{pmatrix}5\\0\end{pmatrix}$ and $\overrightarrow{PS} = \begin{pmatrix}2\\3\end{pmatrix}$.

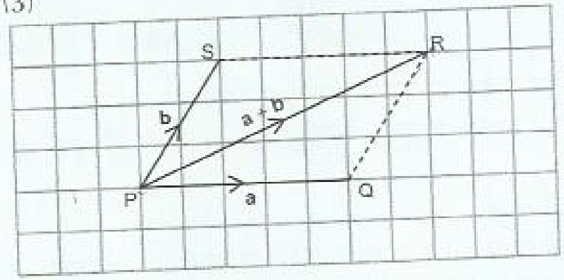

Figure 4.11

The movement of $\begin{pmatrix}5\\0\end{pmatrix}$ is followed by a movement of $\begin{pmatrix}2\\3\end{pmatrix}$.

So $\begin{pmatrix}5\\0\end{pmatrix} + \begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}5+2\\0+3\end{pmatrix} = \begin{pmatrix}7\\3\end{pmatrix}$

$\therefore \overrightarrow{PR} = \begin{pmatrix}7\\3\end{pmatrix}$.

When adding vectors: $\begin{pmatrix}\mathbf{a}\\\mathbf{b}\end{pmatrix} + \begin{pmatrix}\mathbf{c}\\\mathbf{d}\end{pmatrix} = \begin{pmatrix}\mathbf{a}+\mathbf{c}\\\mathbf{b}+\mathbf{d}\end{pmatrix}$

Subtract vectors

Subtracting vectors using diagrams

If **a** and **b** are two vectors, we define the subtraction of vectors by:

$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$

To subtract a vector, we add its negative.
In diagrams, we can then add the vector by placing it head-to-tail as in usual vector addition.

Consider the follow

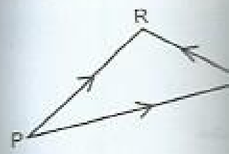

Figure 4.12a

In Fig. 4.12a
$\overrightarrow{PR} - \overrightarrow{PQ} = \overrightarrow{PR} + \overrightarrow{Q}$
$= \overrightarrow{QP} + \overrightarrow{P}$
$= \overrightarrow{QR}$

Fig. 4.13 shows th

Subtracting

To subtract vecto

$\begin{pmatrix}3\\3\end{pmatrix} - \begin{pmatrix}4\\6\end{pmatrix} = \begin{pmatrix}3-4\\3-6\end{pmatrix}$

The zero ve

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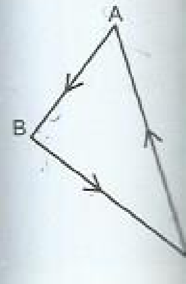


Figure 4.15

Consider the following results:

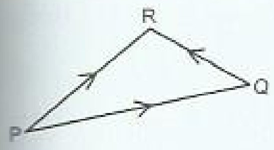


Figure 4.12a

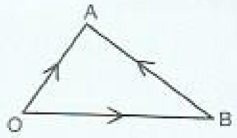


Figure 4.12b

In Fig. 4.12a

$\overrightarrow{PR} - \overrightarrow{PQ} = \overrightarrow{PR} + \overrightarrow{QP}$ (Add negative of PQ)

$= \overrightarrow{QP} + \overrightarrow{PR}$ (Nose-to-tail)

$= \overrightarrow{QR}$

In Fig. 4.12b

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \overrightarrow{OB} + \overrightarrow{AO}$

Fig. 4.13 shows the vector diagram of **a** – **b** using the triangle method.

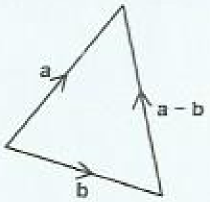


Figure 4.13

Subtracting vectors using coordinates

To subtract vectors we simply subtract the coordinates.

$\begin{pmatrix}3\\3\end{pmatrix} - \begin{pmatrix}4\\6\end{pmatrix} = \begin{pmatrix}3-4\\3-6\end{pmatrix} = \begin{pmatrix}-1\\-3\end{pmatrix}$

The zero vector

Imagine travelling from Mansa to Mpika to Serenje and back to Mansa. You would have covered a large distance, but actually your displacement is zero!

The displacement of an object from the starting point to a final point is zero when the added vectors form a closed polygon, as shown in Fig. 4.14.

Figure 4.14

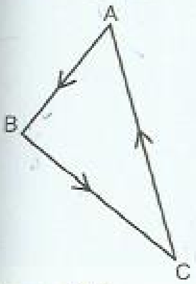


Figure 4.15

$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0$

We denote a zero vector by $0 = \begin{pmatrix}0\\0\end{pmatrix}$, or by $\vec{0}$.

Subtracting a vector from itself is the same as adding the additive inverse or negative of the vector.

$\overrightarrow{AB} + \overrightarrow{BA} = \overrightarrow{AB} - \overrightarrow{AB} = 0$

Example: $\begin{pmatrix}3\\4\end{pmatrix} + \begin{pmatrix}-3\\-4\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

Worked example 2

1 Complete the following.

a) $\overrightarrow{SP} + \overrightarrow{PQ} =$

b) $\overrightarrow{QR} +$ ____ $= \overrightarrow{QS}$

c) $\overrightarrow{QS} + \overrightarrow{SQ} =$ ____

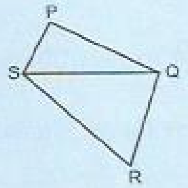


Figure 4.16

2 Simplify.

$\overrightarrow{CA} - \overrightarrow{CB}$

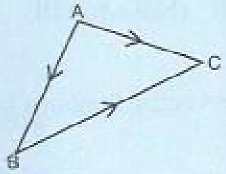


Figure 4.17

3 Use (i) a diagram and (ii) coordinates to calculate the following.

a) $\mathbf{a} + \mathbf{b}$

b) $\mathbf{b} - \mathbf{a}$

c) $\mathbf{a} - \mathbf{b}$

d) $\mathbf{a} + (-\mathbf{a})$

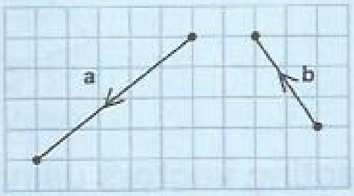


Figure 4.18

Answers

1 Using the head-to-tail method:

a) $\overrightarrow{SP} + \overrightarrow{PQ} = \overrightarrow{SQ}$

b) $\overrightarrow{QR} + \overrightarrow{RS} = \overrightarrow{OS}$

c) $\overrightarrow{QS} + \overrightarrow{SQ} = 0$

2 $\overrightarrow{CA} - \overrightarrow{CB}$

$= \overrightarrow{CA} + \overrightarrow{BC}$

$= \overrightarrow{BC} + \overrightarrow{CA}$

$= \overrightarrow{BA}$

3 a) (i) $\mathbf{a} + \mathbf{b}$

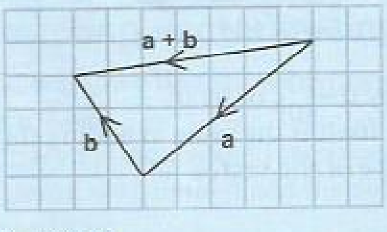


Figure 4.19

(ii) $\begin{pmatrix}-5\\-4\end{pmatrix} + \begin{pmatrix}-2\\3\end{pmatrix} = \begin{pmatrix}-7\\-1\end{pmatrix}$

Worked exam

b) (i) $\mathbf{b} - \mathbf{a} =$

Figure 4.20

c) (i) $\mathbf{a} - \mathbf{b} =$

Figure 4.21

d) (i) $\mathbf{a} + (-\mathbf{a})$

The answ

Activity 2

You will need grid p

1 Look at vectors

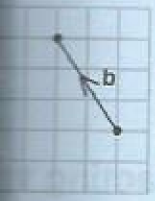
a) Draw ΔXYZ

v and $\overrightarrow{YZ}$ rep

b) Name the di

represents v

c) Show this ad



Worked example 2 (continued)

b) (i) $\mathbf{b} - \mathbf{a} = \mathbf{b} + (-\mathbf{a})$ (ii) $\mathbf{b} - \mathbf{a} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} - \begin{pmatrix} -5 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$

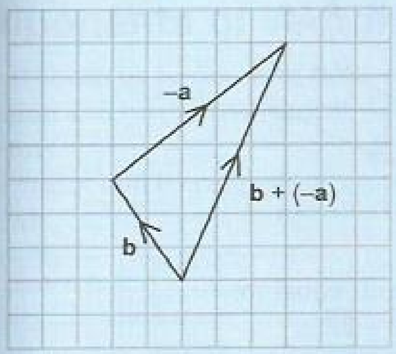


Figure 4.20

c) (i) $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$ (ii) $\mathbf{a} - \mathbf{b} = \begin{pmatrix} -5 \\ -4 \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ -7 \end{pmatrix}$

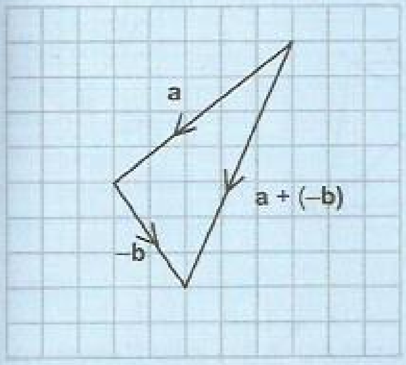


Figure 4.21

d) (i) $\mathbf{a} + (-\mathbf{a}) = \mathbf{a} - \mathbf{a} = 0$ (ii) $\begin{pmatrix} -5 \\ -4 \end{pmatrix} - \begin{pmatrix} -5 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

The answer is the zero vector.

Activity 2

You will need grid paper for this activity.

1 Look at vectors **u** and **v** in Fig. 4.22.

- a) Draw ΔXYZ in which $\overrightarrow{XY}$ represents **v** and $\overrightarrow{YZ}$ represents **u**.
- b) Name the directed line segment that represents **v** + **u**.
- c) Show this addition using coordinates.

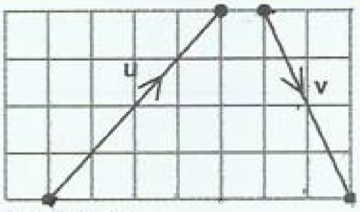
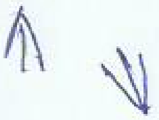


Figure 4.22



Activity 2 (continued)

2 a) Add vectors **a** and **b** in Fig. 4.23 using the head-to-tail method.
b) Use coordinates to check whether **a** + **b** = **b** + **a**.

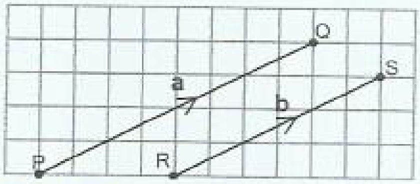


Figure 4.23

3 a) In Fig. 4.24, name directed line segments equal to:
(i) $\overrightarrow{AE} + \overrightarrow{EC}$
(ii) $\overrightarrow{DB} + \overrightarrow{BE}$
(iii) $\overrightarrow{AD} + \overrightarrow{DB} + \overrightarrow{BC}$
(iv) $\overrightarrow{CB} + \overrightarrow{BE} + \overrightarrow{EA} + \overrightarrow{AD}$
b) Copy and complete:
(i) $\overrightarrow{AE}$ + _____ = $\overrightarrow{AB}$
(ii) $\overrightarrow{AD}$ + _____ + $\overrightarrow{EC} = \overrightarrow{AC}$

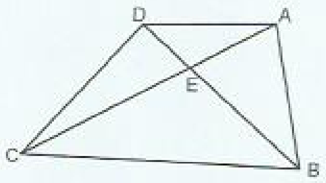


Figure 4.24

4 With the vectors in Fig. 4.25, use the parallelogram method to find:
a) **p** + **q**
b) **p** + **q** + **r**
c) **r** − **q**
d) **p** − **q**
e) **p** − **r** − **q**

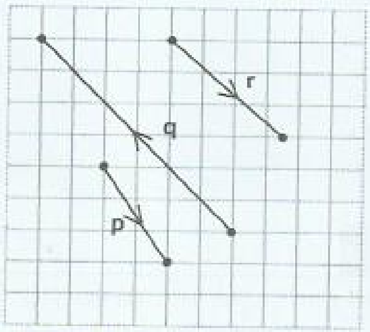


Figure 4.25

Componen

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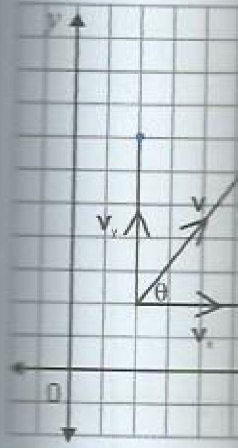
The vector and
g. 4.26b.



igure 4.26a

The magnitu

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We show the len
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The length can b
f Pythagoras.
If $\overrightarrow{PQ} = \binom{3}{4}$, then
In general, if $\overrightarrow{PQ}$

method.

SUB-TOPIC 3 Translations

Components of vectors and the magnitude

Any vector can be broken into two vector components, one in the horizontal x-direction and one in the vertical y-direction. This follows from the addition rule of vectors.

The vector and its components forms a right-angled triangle as shown in Fig. 4.26b.

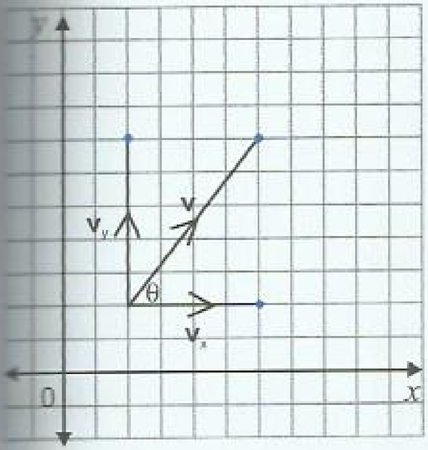

Figure 4.26a

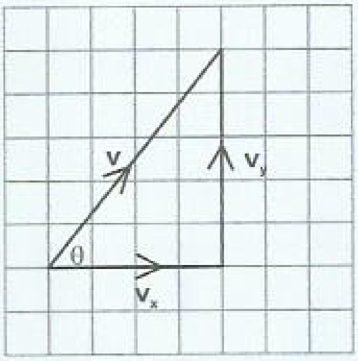

Figure 4.26b

The magnitude of a vector

The magnitude of a vector $\overrightarrow{PQ}$ is the length of the directed line segment PQ. Length is always a non-negative real number.

We show the length of the vector by $|\overrightarrow{PQ}|$. This is called the **modulus** of vector $\overrightarrow{PQ}$.

The length can be derived from the Theorem of Pythagoras.

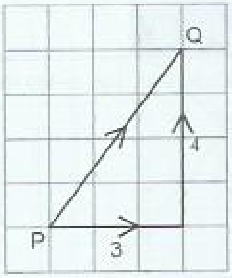

Figure 4.27

If $\overrightarrow{PQ} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, then $|\overrightarrow{PQ}| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$ units

In general, if $\overrightarrow{PQ} = \begin{pmatrix} x \\ y \end{pmatrix}$, then $|\overrightarrow{PQ}| = \sqrt{x^2 + y^2}$ (By Pythagoras)

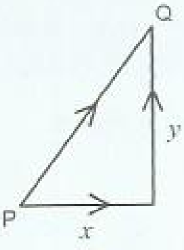

Figure 4.28

Worked example 3

1 In Fig. 4.29, $\overrightarrow{PQ}$, $\overrightarrow{QR}$, $\overrightarrow{RS}$ and $\overrightarrow{ST}$ are vectors.

a) Write each vector using vector coordinates.

b) Express $\overrightarrow{QP}$ in coordinate form.

c) Find $|\overrightarrow{QR}|$.

d) Express the following as single vectors in coordinate form.

(i) $\overrightarrow{RS} + \overrightarrow{ST}$

(ii) $\overrightarrow{RS} - \overrightarrow{ST}$

e) Calculate the magnitude of $\overrightarrow{RS} - \overrightarrow{ST}$.

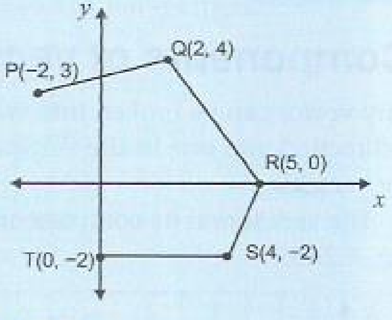

Figure 4.29

Answers

1 a) $\overrightarrow{PQ} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$; $QR = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$; $\overrightarrow{RS} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$ and $\overrightarrow{ST} = \begin{pmatrix} -4 \\ 0 \end{pmatrix}$

b) $\overrightarrow{QP} = -\overrightarrow{PQ} = -\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$

c) $\overrightarrow{QR} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \therefore |\overrightarrow{QR}| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$ units

d) (i) $\overrightarrow{RS} + \overrightarrow{ST} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} + \begin{pmatrix} -4 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ -2 \end{pmatrix}$

(ii) $\overrightarrow{RS} - \overrightarrow{ST} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} - \begin{pmatrix} -4 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

e) We need to calculate $|\overrightarrow{RS} - \overrightarrow{ST}|$ where $\overrightarrow{RS} - \overrightarrow{ST} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

$|\overrightarrow{RS} - \overrightarrow{ST}| = \sqrt{3^2 + (-2)^2} = \sqrt{13}$ units

Activity 3

1 Calculate the magnitude of all the vectors in Fig. 4.30.

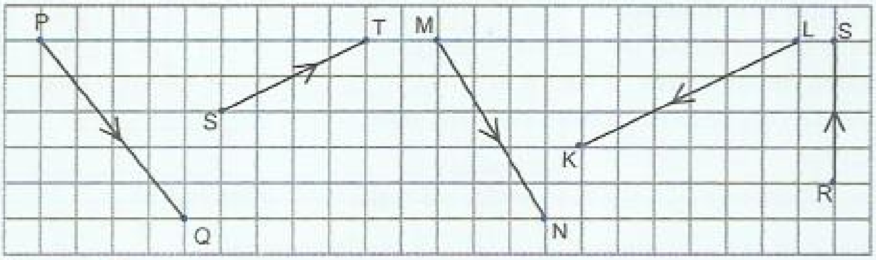

Figure 4.30

2 Given that $\overrightarrow{AB} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$, find:

a) $\overrightarrow{BA}$ b) $|\overrightarrow{AB}|$

3 Calculate the magnitude of each vector.

a) $\begin{pmatrix} -6 \\ 10 \end{pmatrix}$ b) $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ c) $\begin{pmatrix} 13 \\ -2 \end{pmatrix}$ d) $\begin{pmatrix} -3 \\ 7 \end{pmatrix}$

Activity 3 (co

4 In Fig. 4.31, u

a) Write $\overrightarrow{AB}$

b) Find the le

c) Find the d

Figure 4.31

Vectors and

We often use vectors
the Cartesian plane.
free vectors. Any two
length and parallel t
identical. They do no
initial and terminal p

The vectors in Fig
of $\begin{pmatrix} 4 \\ -4 \end{pmatrix}$. We can have
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vector $\begin{pmatrix} 4 \\ -4 \end{pmatrix}$. Only the
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free vector: a vector t
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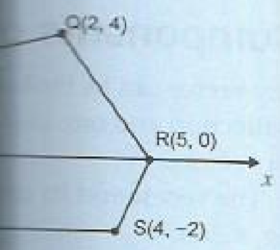

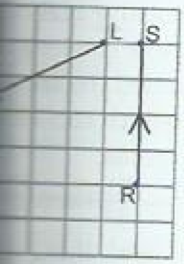

d) $\begin{pmatrix}-3\\7\end{pmatrix}$

Activity 3 (continued)

4 In Fig. 4.31, use vectors to answer the following.
  - a) Write $\overrightarrow{AB} + \overrightarrow{BC}$ in coordinate form
  - b) Find the length of AC.
  - c) Find the distance from C to E.

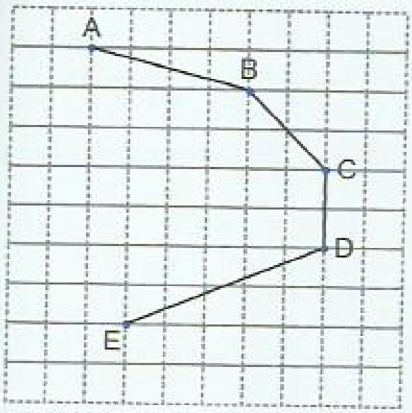

Figure 4.31

Vectors and translations

We often use vectors to show translation on the Cartesian plane. We call these vectors free vectors. Any two vectors of the same length and parallel to each other are considered identical. They do not need to have the same initial and terminal points.

The vectors in Fig. 4.32 all show a translation of $\begin{pmatrix}4\\-4\end{pmatrix}$. We can have an infinite set of parallel line segments representing the same free vector $\begin{pmatrix}4\\-4\end{pmatrix}$. Only the magnitude and direction are given, not the starting point or the point at which the vector acts.

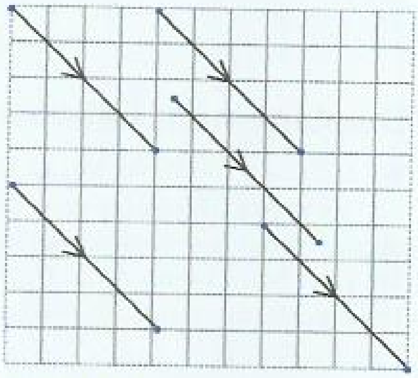

Figure 4.32

New word

free vector: a vector that is used to show a change in position of a point or object on the Cartesian plane.

Position vectors

Each free vector has one corresponding position vector, which is the image of the origin as a result of the same translation. In Fig. 4.33, $\overrightarrow{AB}$ translates A onto B and is the free vector $\begin{pmatrix}6\\4\end{pmatrix}$.

Unlike a free vector, a position vector always begins at the origin O. A position vector describes the position of a point relative to the origin. The vector **p** starts at O and its endpoint is at P. This displacement gives the position of P relative to the origin, so **p** is called the position vector of P. The vector $\mathbf{p} = \begin{pmatrix}6\\4\end{pmatrix}$ and the coordinates of P are (6, 4).

In general, the coordinates of a point P(*x*, *y*) are the components of its position vector $\overrightarrow{OP} = \mathbf{p} = \begin{pmatrix}x\\y\end{pmatrix}$.

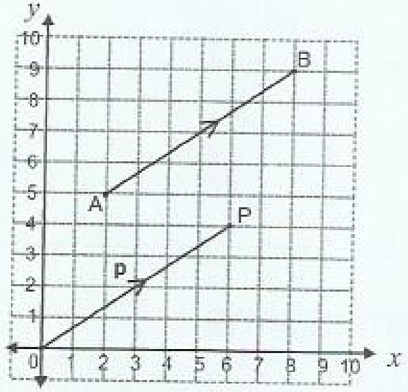

Figure 4.33

Displacement between two vectors

We can use position vectors to find the displacement between two points.

The vector $\overrightarrow{AB}$ represents the displacement between A and B. Note that displacement is always shown by another vector, not by a length.

We can obtain vector $\overrightarrow{AB}$ in Fig. 4.34 as follows:

Draw position vectors for points A and B.

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB}\\ &= -\overrightarrow{OA} + \overrightarrow{OB}\\ &= -\begin{pmatrix}x_1\\y_1\end{pmatrix} + \begin{pmatrix}x_2\\y_2\end{pmatrix}\\ &= \begin{pmatrix}x_2\\y_2\end{pmatrix} - \begin{pmatrix}x_1\\y_1\end{pmatrix}\\ \overrightarrow{AB} &= \begin{pmatrix}x_2-x_1\\y_2-y_1\end{pmatrix}\end{aligned}$$

or

$$\begin{aligned}\overrightarrow{OA} + \overrightarrow{AB} &= \overrightarrow{OB}\\ \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA}\\ &= \begin{pmatrix}x_2\\y_2\end{pmatrix} - \begin{pmatrix}x_1\\y_1\end{pmatrix}\\ \overrightarrow{AB} &= \begin{pmatrix}x_2-x_1\\y_2-y_1\end{pmatrix}\end{aligned}$$

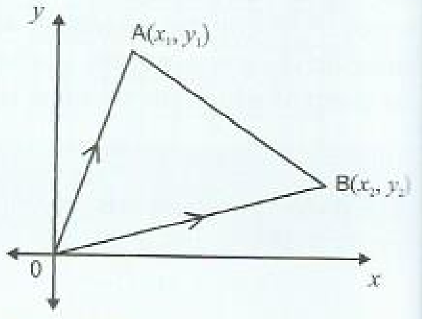

Figure 4.34

Hence if **a** and **b** are the position vectors of points A and B, then $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.

The displacement between two vectors will always be another vector, because displacement has both magnitude and direction.

ch is the image of the
nslates A onto B and i

e origin O. A position
. The vector p starts a
on of P relative to the
$\begin{pmatrix}6\\4\end{pmatrix}$ and the

onents of its position

two points.
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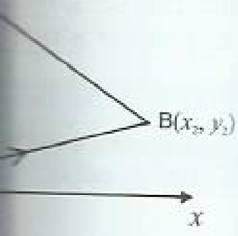
y)



ector, because

We work out the magnitude of the position vector as follows:

$$\overline{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

This result is true for any two points with coordinates A(x_1, y_1) and B(x_2, y_2).

Worked example 4

1 Draw the position vector for each of the following vectors on a Cartesian plane.

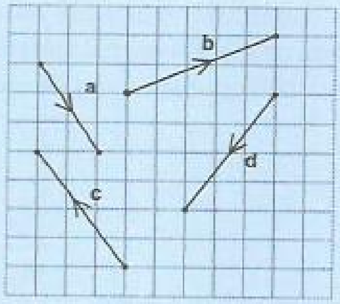


Figure 4.35

2 Write down the coordinates of position vectors **p**, **q** and **r** for $\overrightarrow{AB}$, $\overrightarrow{EF}$ and $\overrightarrow{CD}$.

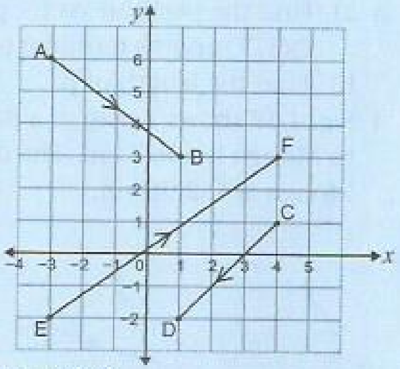


Figure 4.36

Answers

1 Position vectors drawn from the origin.

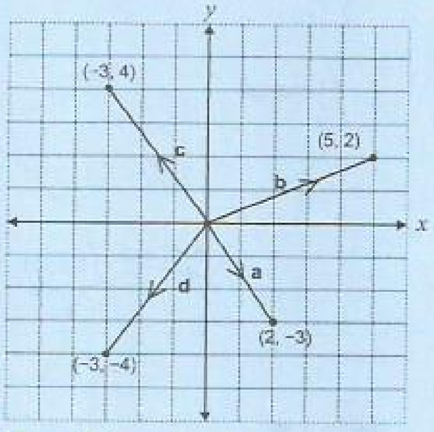


Figure 4.37

2 The coordinates of the position vectors are the coordinates of the vectors.
$\overrightarrow{AB} = \mathbf{p} = \begin{pmatrix}4\\-3\end{pmatrix}$, $\overrightarrow{EF} = \mathbf{q} = \begin{pmatrix}7\\5\end{pmatrix}$
and $\overrightarrow{CD} = \mathbf{r} = \begin{pmatrix}-3\\-3\end{pmatrix}$

Activity 4

1 On grid paper draw position vectors for each vector.

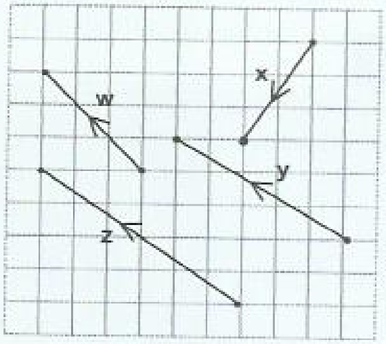


Figure 4.38

2 P is the point (3, 4). $\overrightarrow{PQ} = \begin{pmatrix}-6\\3\end{pmatrix}$. Find the position vector of Q.

3 a) Find the position vector **p** for a vector that has the starting point at Q(3, 7) and its terminal point at P(–4, 2).

b) Find the length of **p**.

4 Fig. 4.39 shows a map of a part of Zambia. The distance by road from Lusaka to Mongu is 606.7 km. The distance of the straight line from Lusaka to Mongu is 552.96 km.

Figure 4.39

Trace this map onto grid paper, with all the towns in the correct positions. Make your drawing as accurate as possible.

Activity 4 (c

a) If Lusak
paper to
b) Now dra
c) Draw a
d) Use the
Mumbw
kilometr

of Q.
tarting point at

by road from Lusaka
from Lusaka to

rrect positions.

Activity 4 (continued)

a) If Lusaka was at the origin, draw a position vector on a separate piece of paper to show the displacement of Kalomo, using the same scale.

b) Now draw a position vector for Mumbwa.

c) Draw a vector to show the displacement from Kalomo to Mumbwa.

d) Use the modulus to calculate the direct distance between Kalomo and Mumbwa. The scale of the drawing is 1 cm : 30 km. Write the distance in kilometres.

Scalar multiplication

When vector **a** is multiplied by a scalar *k*, where *k* represents any number, the result is *k***a**.

The expression *k***a** represents a vector with a magnitude *k* times that of vector **a**. The direction remains the same as in the original vector **a**.

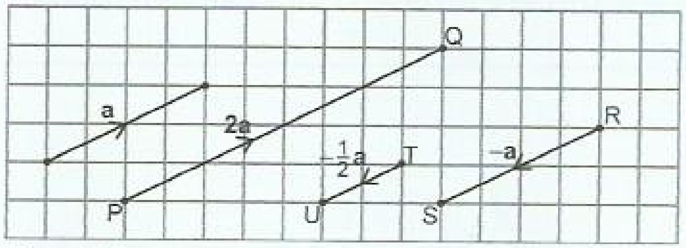


Figure 4.40

In Fig. 4.40, $\overrightarrow{PQ} = 2\mathbf{a} = 2\begin{pmatrix}4\\2\end{pmatrix} = \begin{pmatrix}8\\4\end{pmatrix}$. $\overrightarrow{PQ}$ has the same direction as **a**, but has twice its magnitude. $\overrightarrow{RS} = -\mathbf{a} = -\begin{pmatrix}4\\2\end{pmatrix} = \begin{pmatrix}-4\\-2\end{pmatrix}$

Vectors **a** and **–a** have the same magnitude, but are opposite in direction. The negative sign reverses the direction of the vector.

Worked example 5

1 Given $\mathbf{u} = \begin{pmatrix}2\\1\end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix}-1\\1\end{pmatrix}$

a) Express each of the following as a single vector in coordinate form.
(i) 3**u** (ii) –2**v** (iii) **u** + **v** (iv) 2**u** + 2**v** (v) –2**u** – 2**v**

b) Draw directed line segments on grid paper to represent each vector in a).

Answer

1 a) (i) $3\mathbf{u} = 3\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}6\\3\end{pmatrix}$ (ii) $-2\mathbf{v} = -2\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}2\\-2\end{pmatrix}$

(iii) $\mathbf{u} + \mathbf{v} = \begin{pmatrix}2\\1\end{pmatrix} + \begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$ (iv) $2\mathbf{u} + 2\mathbf{v} = 2\begin{pmatrix}2\\1\end{pmatrix} + 2\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}2\\4\end{pmatrix}$

(v) $-2\mathbf{u} - 2\mathbf{v} = -2\begin{pmatrix}2\\1\end{pmatrix} - 2\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}-2\\-4\end{pmatrix}$

b)

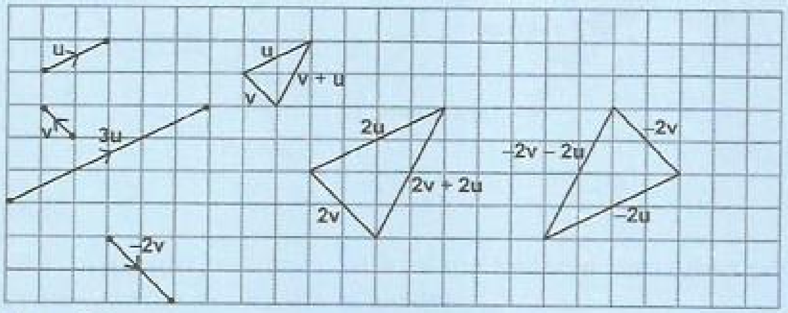


Figure 4.41

Activity 5

Questions 1 to

$\mathbf{p} = \begin{pmatrix}-3\\4\end{pmatrix}$, q =

1 Express eac
a) 7p
e) q – r
i) p – 4q +

2 Calculate th
necessary.
a) |p|
e) |r + p|

3 Is |p + r| = |p

4 Given that
a) |OA|

5 a) Find the
$\begin{pmatrix}3\\9\end{pmatrix}$ – m =
b) Hence fi

6 Copy Fig. 4.
a) Express
b) Given tha
c) Given tha
d) Find:
(i) the co
e) What kin

7 Given that a
a) Express 2a
b) Given tha

Activity 5

Questions 1 to 3 refer to the following vectors:

$\mathbf{p} = \begin{pmatrix}-3\\4\end{pmatrix}$, $\mathbf{q} = \begin{pmatrix}-4\\1\end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix}0\\-6\end{pmatrix}$

1 Express each of the following as a single vector in coordinate notation:

a) $7\mathbf{p}$ b) $-5\mathbf{q}$ c) $\frac{1}{2}\mathbf{r}$ d) $\mathbf{p} + \mathbf{q}$

e) $\mathbf{q} - \mathbf{r}$ f) $\mathbf{r} - \mathbf{q}$ g) $2\mathbf{q} - \mathbf{p}$ h) $\mathbf{r} + 3\mathbf{q}$

i) $\mathbf{p} - 4\mathbf{q} + 3\mathbf{r}$ j) $2\mathbf{p} + \mathbf{q} - 3\mathbf{r}$

2 Calculate the following. Leave your answer in square root form where necessary.

a) $|\mathbf{p}|$ b) $|\mathbf{q}|$ c) $|\mathbf{r}|$ d) $|\mathbf{p} + \mathbf{r}|$

e) $|\mathbf{r} + \mathbf{p}|$ f) $|3\mathbf{p}|$ g) $|\mathbf{r} - \mathbf{p}|$

3 Is $|\mathbf{p} + \mathbf{r}| = |\mathbf{p}| + |\mathbf{r}|$?

4 Given that $\overrightarrow{OA} = \begin{pmatrix}8\\6\end{pmatrix}$, BC = $\begin{pmatrix}x\\9\end{pmatrix}$ and $\overrightarrow{BC} = k\overrightarrow{OA}$, find the values of:

a) |OA| b) k c) x

5 a) Find the vector **m** such that:

$\begin{pmatrix}3\\9\end{pmatrix} - \mathbf{m} = \begin{pmatrix}1\\4\end{pmatrix}$.

b) Hence find $|\mathbf{m}|$.

6 Copy Fig. 4.42 into your exercise book.

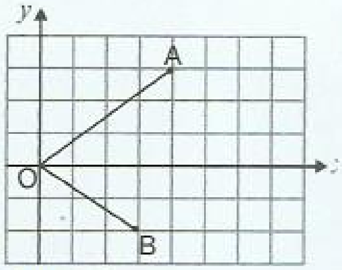


Figure 4.42

a) Express $\overrightarrow{OA}$ and $\overrightarrow{OB}$ in coordinate form.

b) Given that $\overrightarrow{OP} = 2\overrightarrow{OA}$, mark and clearly label P on the diagram.

c) Given that $\overrightarrow{BQ} = 3\overrightarrow{OA}$, mark and clearly label Q on the diagram.

d) Find:

(i) the coordinate form of $\overrightarrow{AQ}$ (ii) $|\overrightarrow{BQ}|$

e) What kind of polygon is figure OPQB? Explain.

7 Given that $\mathbf{a} = \begin{pmatrix}1\\-4\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}m\\2\end{pmatrix}$:

a) Express $2\mathbf{a} - \mathbf{b}$ as a single vector.

b) Given that **a** is parallel to **b**, find the value of m.

SUB-TOPIC 5 Collinearity

Determine whether points are collinear

New word

collinear: lying on the same straight line

If the points P, Q and R are collinear, then $\overrightarrow{PQ} = k\overrightarrow{QR}$. The vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ have the same gradient, and they also share a point at Q.

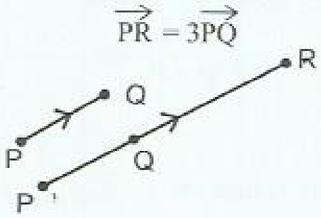


Figure 4.43

$\overrightarrow{PQ} : \overrightarrow{QR} = 1 : 2$
$\overrightarrow{PR}$ is 3 times as long as $\overrightarrow{PQ}$, and has the same direction.

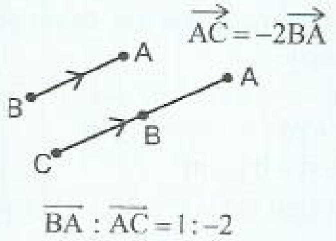


Figure 4.44

$\overrightarrow{BA} : \overrightarrow{AC} = 1 : -2$
$\overrightarrow{AC}$ is twice as long as $\overrightarrow{BA}$, but is in the opposite direction.

Worked example 6

1 $\overrightarrow{OP} = \mathbf{p}$, $\overrightarrow{OQ} = \mathbf{q}$ and point M lies on $\overrightarrow{PQ}$ such that $\overrightarrow{PM} : \overrightarrow{MQ} = 2 : 5$. Express $\overrightarrow{OM}$ in terms of **p** and **q**.

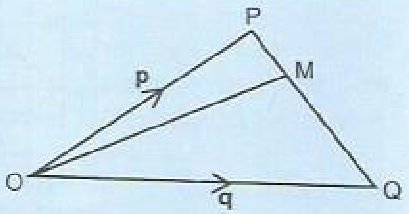


Figure 4.45

2 $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$.
$\overrightarrow{OP} = 3\overrightarrow{OA}$ and $\overrightarrow{OQ} = 2\overrightarrow{OB}$.
R is a point on $\overrightarrow{PQ}$ produced where $\overrightarrow{PQ} = \overrightarrow{QR}$.

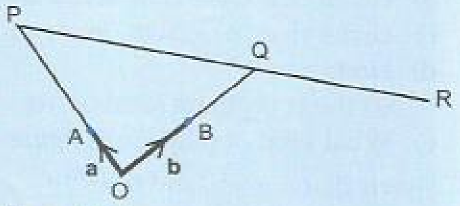


Figure 4.46

a) Express the following in terms of a and/or **b**:
  (i) $\overrightarrow{PO}$ (ii) $\overrightarrow{PQ}$
  (iii) $\overrightarrow{AB}$ (iv) $\overrightarrow{AR}$

b) Write down two facts about the points A, B and R from your answers to Question 2a (iii) and (iv).

ear

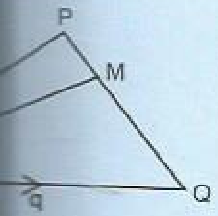
New word

r: lying on the same
ine

$\overrightarrow{AC} = -2\overrightarrow{BA}$

-2

ong as $\overrightarrow{BA}$, but is in
ection.



n your answers to

Worked example 6 (continued)

Answers

1 $\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$ (using ΔOPQ)

$= -\overrightarrow{OP} + \overrightarrow{OQ}$

$= -\mathbf{p} + \mathbf{q}$

$\overrightarrow{PQ} = \mathbf{q} - \mathbf{p}$

$\overrightarrow{OM} = \overrightarrow{OP} + \overrightarrow{PM}$ (using ΔOPM)

$= \mathbf{p} + \frac{2}{7}\overrightarrow{PQ}$

$= \mathbf{p} + \frac{2}{7}(\mathbf{q} - \mathbf{p})$

$= \mathbf{p} + \frac{2}{7}\mathbf{q} - \frac{2}{7}\mathbf{p}$

$\overrightarrow{OM} = \frac{5}{7}\mathbf{p} + \frac{2}{7}\mathbf{q}$

2 a) (i) $\overrightarrow{PO} = -\overrightarrow{OP}$

$= -3\overrightarrow{OA} = -3\mathbf{a}$

(ii) $\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$

$= \overrightarrow{PO} + 2\overrightarrow{OB}$

$= -3\mathbf{a} + 2\mathbf{b}$

(iii) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \mathbf{b} - \mathbf{a}$

(iv) $\overrightarrow{AR} = \overrightarrow{AP} + \overrightarrow{PR}$

$= \frac{2}{3}\overrightarrow{OP} + 2\overrightarrow{PQ}$

$= 2\mathbf{a} - 6\mathbf{a} + 4\mathbf{b}$

$= 4\mathbf{b} - 4\mathbf{a}$

b) $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

$\overrightarrow{AR} = 4(\mathbf{b} - \mathbf{a})$

(i) A, B and R are collinear.

(ii) $\overrightarrow{AR} = 4\overrightarrow{AB}$, and so the multiples of **a** and **b** in the two vectors are in the same ratio.

Activity 6

1 Express each vector in terms of **a** and/or **b**.

a) $\overrightarrow{AC}$ b) $\overrightarrow{CA}$

c) $\overrightarrow{CD}$ d) $\overrightarrow{DA}$

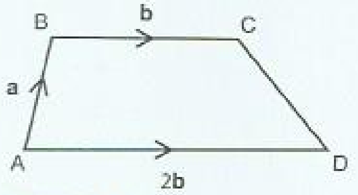


Figure 4.47

Activity 6 (continued)

2 $\overrightarrow{OP} = \mathbf{p}$ and $\overrightarrow{OQ} = \mathbf{q}$. If M is the midpoint of PQ, express $\overrightarrow{OM}$ in terms of **p** and/or **q**.

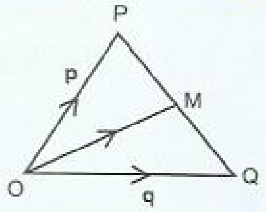


Figure 4.48

3 Fig. 4.49 OACB is a parallelogram, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{AC} = 5\overrightarrow{AD}$.
Express the following in terms of **a** and/or **b**.

a) $\overrightarrow{BC}$
b) $\overrightarrow{CA}$
c) $\overrightarrow{DA}$
d) $\overrightarrow{OD}$
e) $\overrightarrow{BD}$

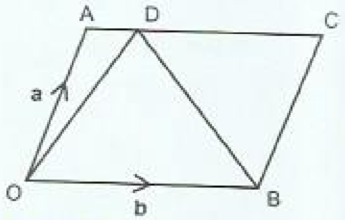


Figure 4.49

4 In Fig. 4.50, A and T are the mid-points of OP and PQ. Given that $\overrightarrow{BQ} = 2\overrightarrow{OB}$, express the following vectors in terms of **a** and **b**.

a) $\overrightarrow{AB}$
b) $\overrightarrow{OQ}$
c) $\overrightarrow{AP}$
d) $\overrightarrow{PQ}$
e) $\overrightarrow{BP}$
f) $\overrightarrow{QA}$
g) $\overrightarrow{OT}$

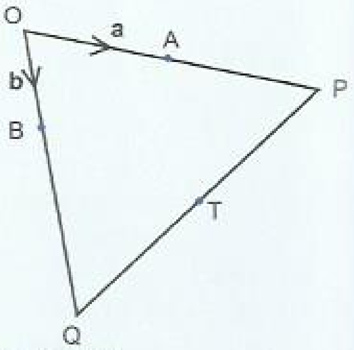
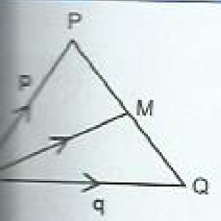
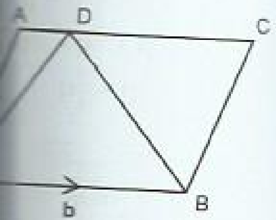


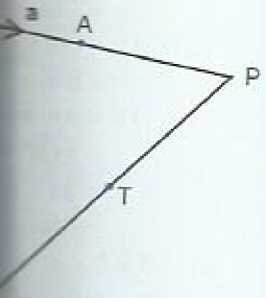
Figure 4.50



re 4.48



4.49



50

SUB-TOPIC 6 Vector geometry

We can use what we have learnt about vectors to solve geometrical problems.

Worked example 7

1 $\overrightarrow{AO} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{ON} = \frac{1}{3}\text{OB}$ and AP : PB = 3 : 2.

a) Express the following in terms of **a** and/or **b**.

(i) $\overrightarrow{AB}$ (ii) $\overrightarrow{OP}$

(iii) $\overrightarrow{AN}$

b) Given that $\overrightarrow{OX} = h\overrightarrow{OP}$, express $\overrightarrow{OX}$ in terms of **a**, **b** and h.

c) Given also that $\overrightarrow{AX} = k\overrightarrow{AN}$, express $\overrightarrow{OX}$ in terms of **a**, **b** and k.

d) Use your answers to Questions 1b and 1c to find the values of h and k.

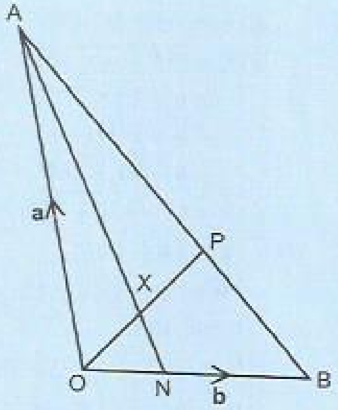


Figure 4.51

2. a) Express $\overrightarrow{AB}$ in terms of **a** and **b**.

b) Given that $\overrightarrow{AP} = h\overrightarrow{AB}$, show that $\overrightarrow{OP} = h\mathbf{b} + (1 - h)\mathbf{a}$.

c) If $\overrightarrow{OC} = 3\mathbf{b}$ and $\overrightarrow{CD} = 2\mathbf{a}$ write down an expression for $\overrightarrow{OD}$ in terms of **a** and **b**.

d) Given that $\overrightarrow{OP} = k\overrightarrow{OD}$, use your answers to Questions 2b and 2c to find the values of h and k.

e) Find the numerical value of the ratio $\frac{\overrightarrow{BP}}{\text{PA}}$.

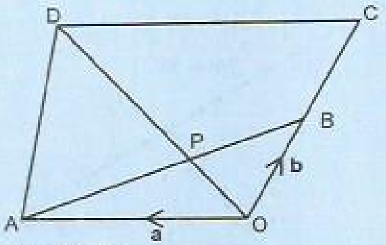


Figure 4.52

Answers

1 a) (i) $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$= -\overrightarrow{OA} + \overrightarrow{OB}$

$= \mathbf{b} - \mathbf{a}$

(ii) $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$ (using ΔOAP)

$= \overrightarrow{OA} + \frac{3}{5}\overrightarrow{AB}$

$= \mathbf{a} + \frac{3}{5}(\mathbf{b} - \mathbf{a})$

$= \frac{2}{5}\mathbf{a} + \frac{3}{5}\mathbf{b}$

(iii) $\overrightarrow{AN} = \overrightarrow{AO} + \overrightarrow{ON}$ (using ΔOAN)

$= -\overrightarrow{OA} + \frac{1}{3}\overrightarrow{OB}$

$= -\mathbf{a} + \frac{1}{3}\mathbf{b}$

$= \frac{1}{3}\mathbf{b} - \mathbf{a}$

Worked example 7 (continued)

b) $\overrightarrow{OX} = h\overrightarrow{OP}$

$= \frac{2}{5}h\mathbf{a} + \frac{3}{5}h\mathbf{b}$

c) $\overrightarrow{AX} = k\overrightarrow{AN}$

$\overrightarrow{AX} = k(\frac{1}{3}\mathbf{b} - \mathbf{a})$

$\overrightarrow{OX}$ (using ΔOAX)

$\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX}$

$= \mathbf{a} + k\overrightarrow{AN}$

$= \mathbf{a} + k(\frac{1}{3}\mathbf{b} - \mathbf{a})$

$= \mathbf{a} - k\mathbf{a} + \frac{1}{3}k\mathbf{b}$

$\overrightarrow{OX} = (1 - k)\mathbf{a} + \frac{1}{3}k\mathbf{b}$

d) $\overrightarrow{OX} = \frac{2}{5}h\mathbf{a} + \frac{3}{5}h\mathbf{b}$

$\therefore$ we can equate the scalars

Therefore:

$\frac{2}{5}h = 1 - k$ (scalars for **a**)

$\frac{3}{5}h = \frac{1}{3}k$ (scalars for **b**)

Solve the simultaneous equations:

$5(\frac{2}{5}h + k) = 5(1)$

$15(\frac{3}{5}h - \frac{1}{3}k) = 15(0)$

$$\begin{array}{r} 2h + 5k = 5 \\ \underline{+9h - 5k = 0} \\ 11h \qquad = 5 \end{array}$$

$\therefore h = \frac{5}{11}$

and $\frac{2}{\cancel{5}} \times \frac{\cancel{5}}{11} + k = 1$

$\frac{2}{11} + k = 1$

$\therefore k = 1 - \frac{2}{11} = \frac{9}{11}$

2 a) $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$= -\mathbf{a} + \mathbf{b}$

$= \mathbf{b} - \mathbf{a}$

b) $\overrightarrow{AP} = \mathrm{h}\overrightarrow{AB}$

$= h(\mathbf{b} - \mathbf{a})$

$= h\mathbf{b} - h\mathbf{a}$

$\therefore \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$ (using ΔOAP)

$= \mathbf{a} + h\overrightarrow{AB}$

$= \mathbf{a} + h(\mathbf{b} - \mathbf{a})$

$= \mathbf{a} - h\mathbf{a} + h\mathbf{b}$

$\overrightarrow{OP} = (1 - h)\mathbf{a} + h\mathbf{b}$

Worked exa

c) $\overrightarrow{OC} = 3$b

$\therefore \overrightarrow{OD} =$

$=$

d) $\overrightarrow{OP} =$ kO

$= k(3$

$= 3k$

Therefor

$\overrightarrow{OP} =$

$\overrightarrow{OP} =$

$(1 - h) =$

$h =$

Solving s

$-h - 2k$

$\underline{+ \ h - 3k}$

$-5k$

k

But $h = 3$

$\therefore h = 3$

$h = \frac{3}{5}$

e) Numerica

$\overrightarrow{AP} = h\mathbf{b} -$

but $h = \frac{3}{5}$

$\therefore \overrightarrow{AP} = \frac{3}{5}\mathbf{b}$

$\overrightarrow{PA} = -\overrightarrow{AP}$

$= -(\frac{3}{5}\mathbf{b}$

$\overrightarrow{PA} = \frac{3}{5}\mathbf{a} -$

$\overrightarrow{AP} + \overrightarrow{PB} =$

$\overrightarrow{PB} =$

$=$

$=$

$\therefore \overrightarrow{BP} =$

$=$

$=$

$\therefore \frac{\overrightarrow{BP}}{\overrightarrow{PA}} = \frac{\frac{2}{5}\mathbf{a}}{\frac{3}{5}\mathbf{a}}$

$\therefore \overrightarrow{BP} : \overrightarrow{PA}$

Worked example 7 (continued)

c) $\overrightarrow{OC} = 3\mathbf{b}$ and $\overrightarrow{CD} = 2\mathbf{a}$

$\therefore \overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD}$

$= 3\mathbf{b} + 2\mathbf{a}$

d) $\overrightarrow{OP} = \text{k}\overrightarrow{OD}$

$= k(3\mathbf{b} + 2\mathbf{a})$

$= 3k\mathbf{b} + 2k\mathbf{a}$

Therefore:

$\overrightarrow{OP} = (1 - h)\mathbf{a} + h\mathbf{b}$

$\overrightarrow{OP} = 2k\mathbf{a} + 3k\mathbf{b}$

$(1 - h) = 2k$ (scalars for **a**)

$h = 3k$ (scalars for **b**)

Solving simultaneously:

$$\begin{array}{r} -h - 2k = -1 \\ +\ \ h - 3k = 0 \\ \hline -5k = -1 \\ k = \frac{1}{5} \end{array}$$

But $h = 3k$

$\therefore h = 3 \times \frac{1}{5}$

$h = \frac{3}{5}$

e) Numerical ratio for $\frac{\overrightarrow{BP}}{\overrightarrow{PA}}$

$\overrightarrow{AP} = h\mathbf{b} - h\mathbf{a}$

but $h = \frac{3}{5}$

$\therefore \overrightarrow{AP} = \frac{3}{5}\mathbf{b} - \frac{3}{5}\mathbf{a}$

$\overrightarrow{PA} = -\overrightarrow{AP}$

$= -(\frac{3}{5}\mathbf{b} - \frac{3}{5}\mathbf{a})$

$\overrightarrow{PA} = \frac{3}{5}\mathbf{a} - \frac{3}{5}\mathbf{b}$

$\overrightarrow{AP} + \overrightarrow{PB} = \overrightarrow{AB}$

$\overrightarrow{PB} = \overrightarrow{AB} - \overrightarrow{AP}$

$= \mathbf{b} - \mathbf{a} - (\frac{3}{5}\mathbf{b} - \frac{3}{5}\mathbf{a})$

$= \frac{2}{5}\mathbf{b} - \frac{2}{5}\mathbf{a}$

$\therefore \overrightarrow{BP} = -\overrightarrow{PB}$

$= -(\frac{2}{5}\mathbf{b} - \frac{2}{5}\mathbf{a})$

$= \frac{2}{5}\mathbf{a} - \frac{2}{5}\mathbf{b}$

$$\therefore \frac{\overrightarrow{BP}}{\overrightarrow{PA}} = \frac{\frac{2}{5}\mathbf{a} - \frac{2}{5}\mathbf{b}}{\frac{3}{5}\mathbf{a} - \frac{3}{5}\mathbf{b}} = \frac{\frac{2}{5}(\mathbf{a} - \mathbf{b})}{\frac{3}{5}(\mathbf{a} - \mathbf{b})}$$

$\therefore \overrightarrow{BP} : \overrightarrow{PA} = \frac{2}{5} : \frac{3}{5} = 2 : 3$

Activity 7

1 In ΔOAB, P divides $\overrightarrow{AB}$ in the ratio 2 : 3. Find $\overrightarrow{OP}$ in terms of **a** and **b**.

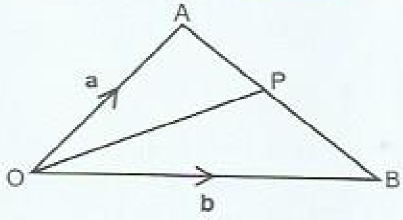


Figure 4.53

2 OCD is a triangle, A is the midpoint of $\overrightarrow{OC}$ and B is the midpoint of $\overrightarrow{OD}$.
$\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$.
Find the following in terms of **a** and **b**.

a) $\overrightarrow{OC}$ b) $\overrightarrow{OD}$
c) **x** d) **y**

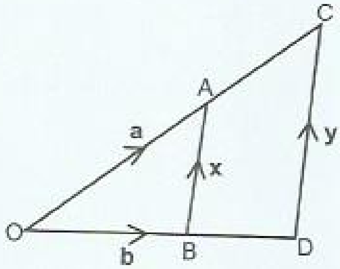


Figure 4.54

3 $\overrightarrow{OA} = 4\mathbf{a}$, $\overrightarrow{AB} = 2\mathbf{b} - 3\mathbf{a}$ and $\overrightarrow{OC} = 3\mathbf{b}$.
Express the following in terms of **a** and **b**.

a) $\overrightarrow{OB}$
b) $\overrightarrow{CB}$

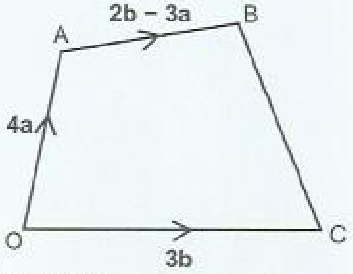


Figure 4.55

4 A is a point on $\overrightarrow{PQ}$, such that $\overrightarrow{QP} = 4\overrightarrow{QA}$. B is the mid-point of $\overrightarrow{OP}$. $\overrightarrow{OA}$ and $\overrightarrow{QB}$ intersect at X.

a) Given that $\overrightarrow{OP} = \mathbf{a}$ and $\overrightarrow{OQ} = \mathbf{b}$, Express the following in terms of **a** and **b**.
   (i) $\overrightarrow{PQ}$ (ii) $\overrightarrow{OA}$ (iii) $\overrightarrow{QB}$
b) If $\overrightarrow{QX} = h\overrightarrow{QB}$, express $\overrightarrow{OX}$ in terms of **a**, **b** and h.
c) If $\overrightarrow{OX} = k\overrightarrow{OA}$, use the answer to Question 4b to find the values of h and k.
d) Hence express $\overrightarrow{OX}$ in terms of **a** and **b** only.

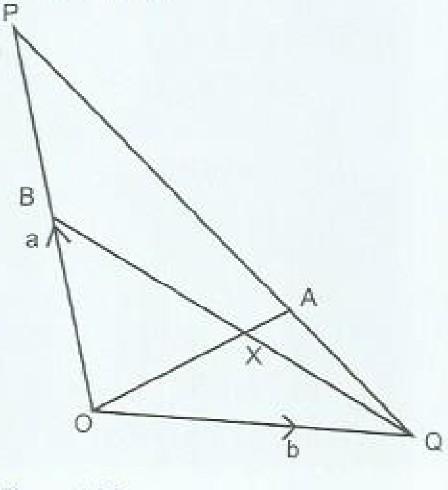


Figure 4.56

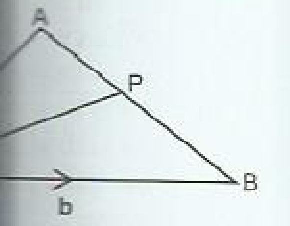
Activity 7 (co

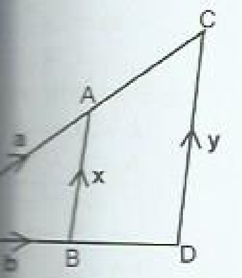
5 OPRQ is a par
positioned suc

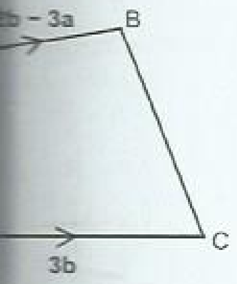
a) Express the
(i) $\overrightarrow{PR}$
b) $\overrightarrow{OR}$ and $\overrightarrow{QE}$
terms of **p**,
(i) $\overrightarrow{QF}$
c) Given also
d) Find the va
e) Find the ra

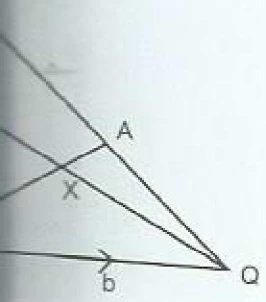
6 In Fig. 4.58, O

a) Express the
(i) $\overrightarrow{AX}$
(ii) $\overrightarrow{OX}$
b) Given that
ratio $\frac{\overline{CX}}{\overline{XO}}$.
c) The vector
expression
d) Given that
equation to
e) Find the nu









Activity 7 (continued)

5 OPRQ is a parallelogram in which $\overrightarrow{OP} = \mathbf{p}$ and $\overrightarrow{OQ} = \mathbf{q}$. The point E on $\overrightarrow{OP}$ is positioned such that $\overrightarrow{OE} : \overrightarrow{EP} = 1 : 2$.

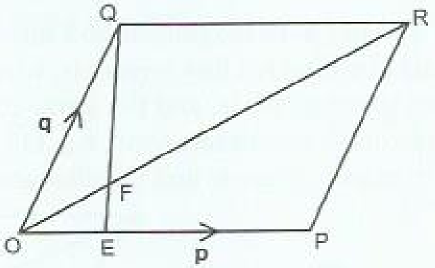


Figure 4.57

a) Express the following in terms of $\mathbf{p}$ and/or $\mathbf{q}$.
   (i) $\overrightarrow{PR}$ (ii) $\overrightarrow{OE}$ (iii) $\overrightarrow{QE}$

b) $\overrightarrow{OR}$ and $\overrightarrow{QE}$ meet at F. Given that $\overrightarrow{QF} = k\overrightarrow{QE}$, express the following in terms of $\mathbf{p}$, $\mathbf{q}$ and k:
   (i) $\overrightarrow{QF}$ (ii) $\overrightarrow{OF}$

c) Given also that $\overrightarrow{OF} = h\overrightarrow{OR}$, express $\overrightarrow{OF}$ in terms of $\mathbf{p}$, $\mathbf{q}$ and h.

d) Find the values of h and k.

e) Find the ratio of $\overrightarrow{OF} : \overrightarrow{OR}$

6 In Fig. 4.58, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{AX} = \overrightarrow{XB}$

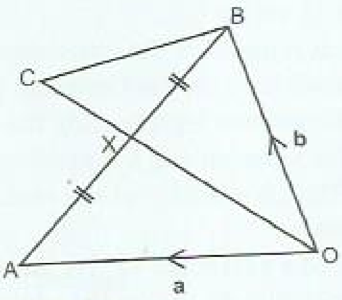


Figure 4.58

a) Express the following in terms of $\mathbf{a}$ and $\mathbf{b}$.
   (i) $\overrightarrow{AX}$
   (ii) $\overrightarrow{OX}$

b) Given that $\overrightarrow{OC} = \frac{3}{4}\mathbf{a} + \frac{3}{4}\mathbf{b}$, write down the numerical value of the ratio $\frac{\overrightarrow{CX}}{\overrightarrow{XO}}$.

c) The vector $\overrightarrow{BC}$ is produced to the point Y, such that $\overrightarrow{BY} = h\overrightarrow{BC}$. Write an expression for $\overrightarrow{BC}$ and $\overrightarrow{BY}$. Hence prove that $\overrightarrow{OY} = \frac{3h}{4}\mathbf{a} - (1 - \frac{h}{4})\mathbf{b}$

d) Given that $\overrightarrow{OY} = k\overrightarrow{OA}$, write an equation involving $\mathbf{a}$, $\mathbf{b}$, h and k. Use this equation to find the values of h and k.

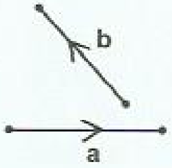
e) Find the numerical value of the ratio $\frac{\overrightarrow{YA}}{\overrightarrow{AO}}$.

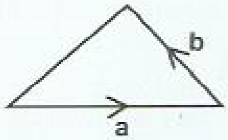
TOPIC

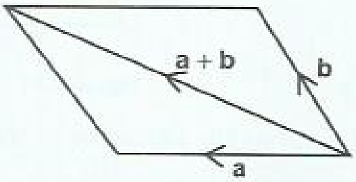
4 Summary, revision and assessment

Summary

- A vector is a quantity that has both magnitude and direction.
- You can represent vectors by directed line segments, where the length of the line segment represents the magnitude, and the arrow represents the direction.
- You can also express vectors in coordinate form, e.g. $\overrightarrow{OA} = \mathbf{b} = \begin{pmatrix} x \\ y \end{pmatrix}$.
- To add vectors, you can use the triangle and parallelogram rules.







- To subtract vectors, add the negative of a vector.
- To multiply a vector by a scalar quantity, you multiply each component of the vector by the scalar. This changes the magnitude but not the direction of the vector.
- Adding a vector and its negative results in the zero vector 0, which has no magnitude and no direction.
- If you arrange directed line segments head to tail to form a closed polygon, then the resultant is the zero vector 0.
- A free vector is a vector that represents the translation of a point to another point and can be shown anywhere on the Cartesian plane. There can be an infinite set of parallel line segments representing the same free vector.
- A position vector is a vector representing a translation by showing the translation of the origin. The coordinates of the endpoint of the vector are the same as the vector coordinates.
- The magnitude (modulus) of a vector $\overrightarrow{AB} = \begin{pmatrix} a \\ b \end{pmatrix}$ is $|\overrightarrow{AB}| = \sqrt{a^2 + b^2}$.
- If the points A, B and C are collinear (lie on the same straight line), then $\overrightarrow{AB} = k\overrightarrow{BC}$. The vectors share the point B.

Revision exercise

1 Given $\mathbf{a} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

a) Express the following as vectors in coordinate form.

(i) $\mathbf{a} + \mathbf{b}$ (ii) $\mathbf{a} - \mathbf{b}$ (iii) $2\mathbf{b} - \mathbf{a}$

b) Find (i) $|\mathbf{a}|$ (ii) $|\mathbf{a} + \mathbf{b}|$ (iii) $|\mathbf{a} - \mathbf{b}|$

2 a) Find the vector $\mathbf{p}$ such that $\begin{pmatrix} 2 \\ -3 \end{pmatrix} + \mathbf{p} = \begin{pmatrix} 7 \\ -4 \end{pmatrix}$

b) Find $|\mathbf{p}|$

---

3 In ΔOPQ, $\overline{PM}$:
$\overrightarrow{OQ} = \mathbf{q}$, find O

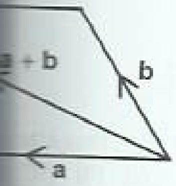
4 HIJK is a trape
If $\overrightarrow{IJ} = \mathbf{a}$ and IH
in terms of a a
a) $\overrightarrow{JH}$
c) $\overrightarrow{KJ}$

5 OPQRST is a re
$\overrightarrow{OP} = \mathbf{a}$ and OR
vectors in term
as possible.
a) $\overrightarrow{RS}$
c) $\overrightarrow{OQ}$

6 Given that p =
a) Find:
(i) $2\mathbf{q} + \mathbf{q}$
b) Given that
to vector r,

7 OPQR is a para
and $\overrightarrow{OR} = \mathbf{a} +$
a) Express th
and/or b, a
(i) $\overrightarrow{QR}$
b) S is a poin
Explain wh

on.
the length of the
esents the direction.
$b = \begin{pmatrix} x \\ y \end{pmatrix}$.
rules.



component of the
direction of the

which has no

losed polygon,

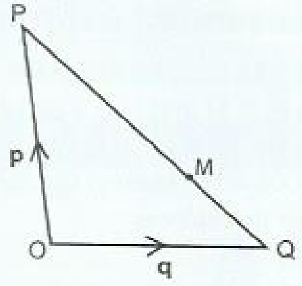
oint to another
here can be an
e vector.
wing the
the vector are the

F.

line), then

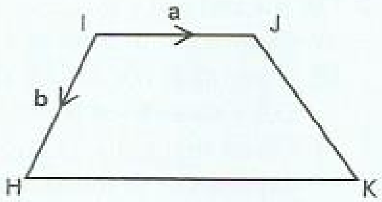
Summary, revision and assessment (continued)

3 In ΔOPQ, $\overrightarrow{PM} : \overrightarrow{MQ} = 3 : 1$. If $\overrightarrow{OP} = \mathbf{p}$ and $\overrightarrow{OQ} = \mathbf{q}$, find $\overrightarrow{OM}$ in terms as **p** and **q**.



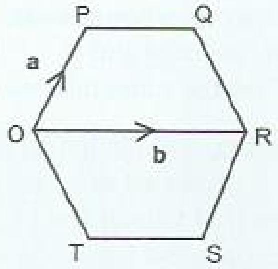
4 HIJK is a trapezium in which $\overrightarrow{HK} = 2\overrightarrow{IJ}$. If $\overrightarrow{IJ} = \mathbf{a}$ and $\overrightarrow{IH} = \mathbf{b}$, express the following in terms of **a** and **b**:

a) $\overrightarrow{JH}$ b) $\overrightarrow{IK}$
c) $\overrightarrow{KJ}$



5 OPQRST is a regular hexagon. Given that $\overrightarrow{OP} = \mathbf{a}$ and $\overrightarrow{OR} = \mathbf{b}$, express the following vectors in terms of **a** and/or **b**, as simply as possible.

a) $\overrightarrow{RS}$ b) $\overrightarrow{PR}$
c) $\overrightarrow{OQ}$

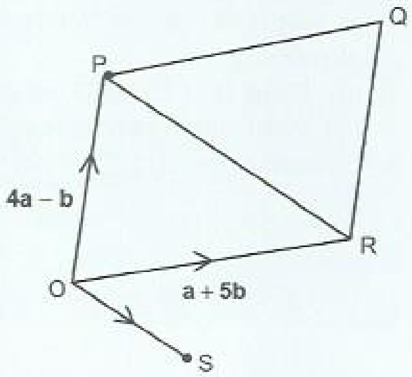


6 Given that $\mathbf{p} = \begin{pmatrix} -8 \\ 6 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} n \\ 15 \end{pmatrix}$.

a) Find:
   (i) $2\mathbf{q} + \mathbf{q}$ (ii) $\mathbf{q} - \mathbf{p}$ (iii) $|\mathbf{p}|$
b) Given that vector **p** is parallel to vector **r**, calculate the value of n.

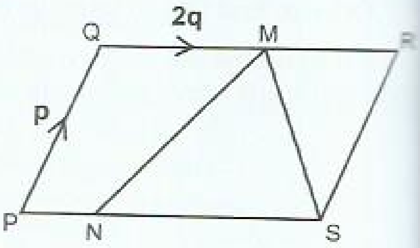
7 OPQR is a parallelogram, $\overrightarrow{OP} = 4\mathbf{a} - \mathbf{b}$ and $\overrightarrow{OR} = \mathbf{a} + 5\mathbf{b}$.

a) Express the following in terms of **a** and/or **b**, as simply as possible.
   (i) $\overrightarrow{QR}$ (ii) $\overrightarrow{PR}$
b) S is a point such that $\overrightarrow{OS} = -\mathbf{p} + 2\mathbf{q}$. Explain why $\overrightarrow{PR}$ is parallel to $\overrightarrow{OS}$.



Summary, revision and assessment (continued)

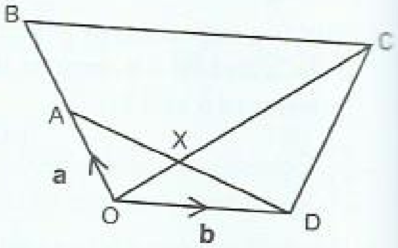
Assessment exercise



1 PQRS is a parallelogram. The midpoint of $\overrightarrow{QR}$ is M and N is a point on $\overrightarrow{PS}$ such that $\overrightarrow{PN} : \overrightarrow{NS} = 1 : 3$. Given that $\overrightarrow{PQ} = \mathbf{p}$ and $\overrightarrow{QM} = 2\mathbf{q}$, express the following in terms of **p** and/or **q**.

a) $\overrightarrow{MS}$ b) $\overrightarrow{SN}$

c) $\overrightarrow{NM}$



2 $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OD} = \mathbf{b}$.

a) Express $\overrightarrow{DA}$ in term of **a** and **b**.

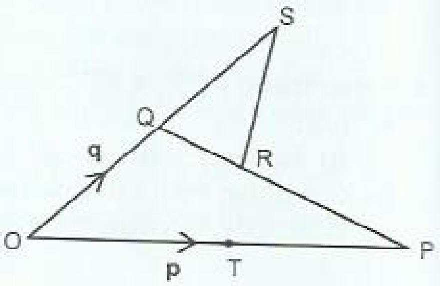
b) Given that $\overrightarrow{DX} = n\overrightarrow{DA}$, show that $\overrightarrow{OX} = n\mathbf{a} + (1 - n)\mathbf{b}$.

c) Given that $\overrightarrow{OB} = 3\mathbf{a}$ and $\overrightarrow{BC} = 2\mathbf{b}$, express $\overrightarrow{OC}$ in terms of **a** and **b**.

d) Given that $\overrightarrow{OX} = m\overrightarrow{OC}$ use your results from Questions 2b and 2c to find the values of *m* and *n*.

e) Find the numerical value of the ratio $\frac{\overrightarrow{AX}}{\overrightarrow{XD}}$.

3 In ΔOPQ, R is a point on $\overrightarrow{PQ}$ such that $\overrightarrow{PR} : \overrightarrow{RQ} = 2 : 1$. $\overrightarrow{OQ}$ is produced to S such that $\overrightarrow{OQ} : \overrightarrow{QS} = 3 : 2$. Given that $\overrightarrow{OP} = \mathbf{p}$ and $\overrightarrow{OQ} = \mathbf{q}$.



a) Express the following vectors in terms of **p** and **q**:

(i) $\overrightarrow{PQ}$ (ii) $\overrightarrow{PR}$

(iii) $\overrightarrow{OR}$ (iv) $\overrightarrow{OS}$

b) Show that $\overrightarrow{RS} = \mathbf{q} - \frac{1}{3}\mathbf{p}$.

c) Given that T is a point on $\overrightarrow{OT}$ such that $\overrightarrow{OT} = \frac{5}{9}\mathbf{p}$, express the vector $\overrightarrow{TS}$ as simply as possible in terms of **p** and **q**.

d) (i) Show that $\overrightarrow{TS} = k\overrightarrow{RS}$, where *k* is a constant.

(ii) Write down two facts about $\overrightarrow{TS}$ and $\overrightarrow{RS}$.

e) Calculate the ratio $\frac{\text{area of } \Delta PTR}{\text{area of } \Delta OTR}$.

OPIC
5

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Stretch
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Combined trans
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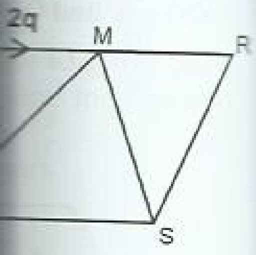
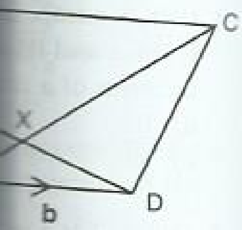
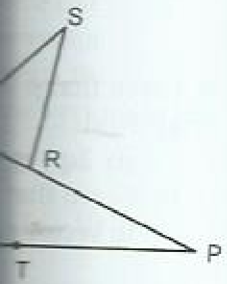






TOPIC

5

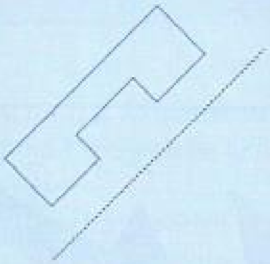
Geometric transformations

Sub-topic	Specific Outcomes
Introduction to transformation	• Introduction to transformation.
Translation	• Use a column vector to translate an object.
Reflection	• Reflect object by different methods.
Rotation	• Rotate object by different methods.
Enlargement	• Enlarge object by different methods.
Stretch	• Stretch object by different methods.
Shear	• Shear objects by different methods.
Combined transformations	• Solve problems involving combined transformation.
Find area scale factors of a stretch by the determinant method.	• Find area scale factors of a stretch by the determinant method.

Starter activity

1 The broken line is the mirror line. Copy and draw the position of the image in each of the following:

a)



b)

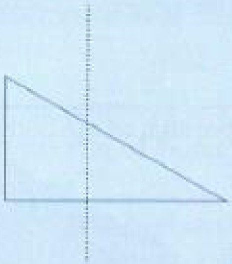


Figure 5.1

2 Fig. 5.2 shows a pathway from P to Q.

a) How many steps downwards and then to the left should Malama take to walk from P to Q?

b) Express PQ as a column vector.

c) Find the column vector that can take Malama from Q back to P.

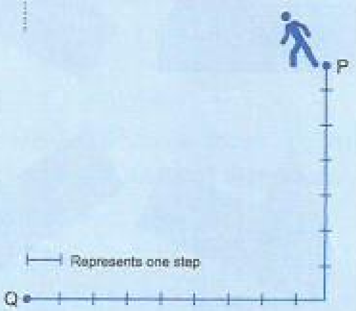


Figure 5.2

SUB-TOPIC 1 Introduction to transformation

A transformation is a geometrical operation which changes the position and/or the shape of an object. The original shape is called the object and the transformed shape is called the image.

Mathematically, we define a shape as the set of points that defines an object. In transformational geometry these are usually described on the Cartesian plane.

In this topic we learn more about these forms of transformations:

- Translation
- Reflection
- Rotation
- Enlargement
- Stretch
- Shear
- Combined transformations.

Only translations, reflections and rotations are examples of congruent (isometric) transformations.

New words

transformation: a change that maps a set of points defining an object onto a set of points defining its image
congruent/isometric: a description of a transformation that keeps the original shape and dimensions of the object.

Activity 1

1 Which of these shapes are congruent?

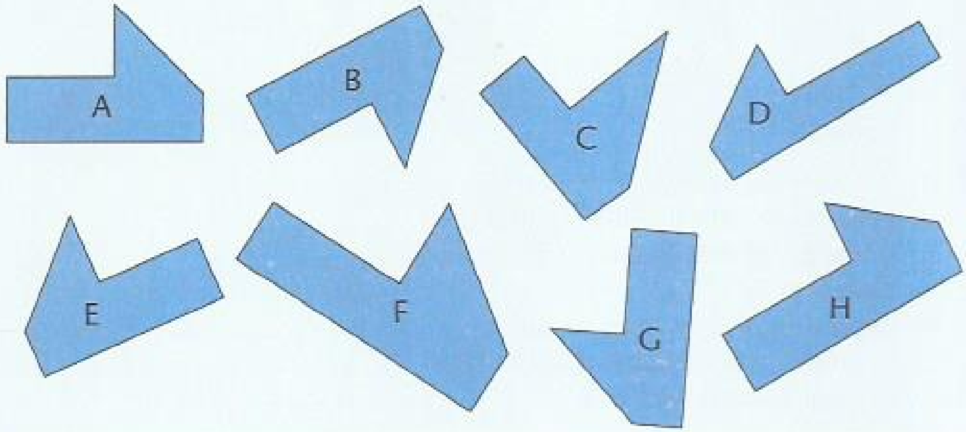


Figure 5.3

UB-TOPIC 2

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ABC onto ΔA'B'
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We can use a
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1 What is the
2 ΔABC with
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Cartesian plane.

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he original shape

SUB-TOPIC 2 Translation

A translation is a transformation which moves (shifts) every point of the object in a straight line into a new position.

Translation is usually denoted by the matrix T.

In a translation, no point stays where it is; every point on the object moves by a fixed distance in the same direction.

Translation is considered to be a direct isometry because the translated image is identical to the object.

Translations represented by column vectors

The figure below shows ΔABC and its image ΔA'B'C' after a translation.

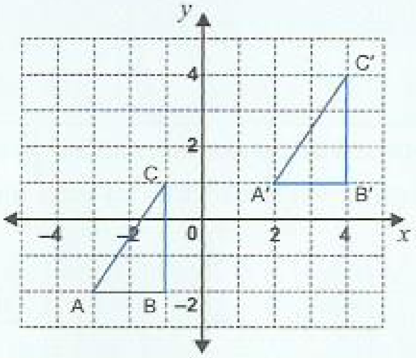

Figure 5.4

The translation is 5 squares to the right and 3 squares upwards. The translation of ΔABC onto ΔA'B'C' can be expressed as a column vector or vector in coordinate form: $T = \begin{pmatrix}5\\3\end{pmatrix}$.

Remember that a vector in coordinate form gives the horizontal (x) change as the top coordinate, and the vertical (y) change as the bottom coordinate.

A column vector is also called a column matrix in the 2 × 1 format.

We can use a column vector to describe any translation, because there are only two possible movements: in the horizontal direction and the vertical direction.

Worked example 1

1 What is the image of the point P(2, 4) under the translation $T = \begin{pmatrix}3\\5\end{pmatrix}$?

2 ΔABC with vertices A(3, 4), B(4, 2) and C(6, 3) is translated by the same column vector. The image of point B is B'(–2, 3). Find

- a) the column vector for the translation matrix
- b) the images of A and C.

Worked example 1 (continued)

Answers

1 Column vector of P + translation matrix = image.

$\begin{pmatrix}2\\4\end{pmatrix}+\begin{pmatrix}3\\5\end{pmatrix}=\begin{pmatrix}5\\9\end{pmatrix}$

The image of P is P'(5, 9).

2 a) Column vector of B + translation matrix = image

Let the translation matrix T be $\begin{pmatrix}a\\b\end{pmatrix}$.

$\begin{pmatrix}4\\2\end{pmatrix}+\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}-2\\3\end{pmatrix}$

$\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}-2\\3\end{pmatrix}-\begin{pmatrix}4\\2\end{pmatrix}=\begin{pmatrix}-6\\1\end{pmatrix}$

b) A + T = A'

$\begin{pmatrix}3\\4\end{pmatrix}+\begin{pmatrix}-6\\1\end{pmatrix}=\begin{pmatrix}-3\\5\end{pmatrix}$

C + T = C'

$\begin{pmatrix}6\\3\end{pmatrix}+\begin{pmatrix}-6\\1\end{pmatrix}=\begin{pmatrix}0\\4\end{pmatrix}$

The images of A and C are A'(–3, 5) and C'(0, 4)

Activity 2

1 Find the image of A(–1, 5) under the translation $\begin{pmatrix}3\\4\end{pmatrix}$.

2 Copy Fig. 5.5 and write down the vectors for each translation.

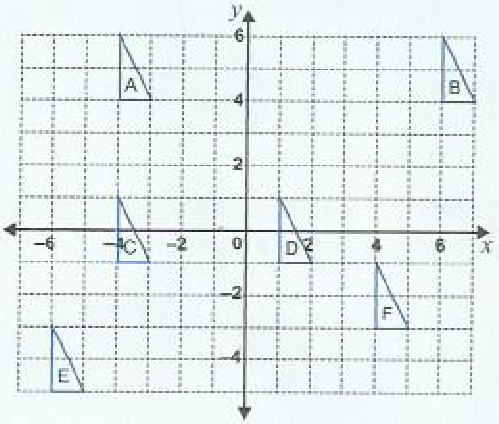

Figure 5.5

a) A onto D	b) B onto C	c) D onto F
d) A onto B	e) D onto E	f) D onto C

3 G'H' is the image of line GH after a translation of $\begin{pmatrix}3\\-6\end{pmatrix}$. If the coordinates of H' are (4, –1), calculate the coordinates of H.

Summary

A translation is a transformation that shifts every point of the object in a straight line into the new position. To describe a translation fully, we give the column vector.

Transforma

Translation is the
transformations a

Remember

A matrix is a rectan
We call this particu

To find the image
$Q=\begin{pmatrix}a & b\\c & d\end{pmatrix}$, we nee
That is, if $A=\begin{pmatrix}x\\y\end{pmatrix}$ a
$\therefore$ A' = $(ax + by,$

Worked exan

Find the image of

Answer

$A'=\begin{pmatrix}2 & 1\\1 & -3\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix}=$

For all transforma
invariant (fixed).
That is, $\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix}$
We can find the 2
(1, 0) and (0, 1).
Given the matrix

Note

If the point (1, 0) is
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Worked exan

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the transformatio

Answer

(1, 0) is mapped o
(0, 1) is mapped o
Hence the transfo

Transformations represented by 2 × 2 matrices

Translation is the only transformation described by a column vector. All the other transformations are described by 2 × 2 matrices.

Remember

A matrix is a rectangular array of numbers written in the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
We call this particular matrix a 2 × 2 matrix.

To find the image $A'(x, y)$ of a point $A(x, y)$ under a transformation matrix $Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we need to multiply A by Q.

That is, if $A = \begin{pmatrix} x \\ y \end{pmatrix}$ and $Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $A' = Q\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$

$\therefore A' = (ax + by, cx + dy)$

Worked example 2

Find the image of A(1, 2) and B(2, –3) under the transformation matrix $Q = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}$

Answer

$A' = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \end{pmatrix}$ $\therefore$ A' is (4, –5). $B' = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 11 \end{pmatrix}$ $\therefore$ B' is (1, 11).

For all transformation matrices represented by 2 × 2 matrices, the origin O′(0, 0) is invariant (fixed).

That is, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\therefore$ O′ is (0, 0)

We can find the 2 × 2 transformation matrix if we know the images of the points (1, 0) and (0, 1).

Given the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then,

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$ First column of the matrix.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$ Second column of the matrix.

Note

If the point (1, 0) is mapped onto (a, c) and (0, 1) is mapped onto (b, d), then the transformation matrix is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Worked example 3

If a transformation maps (1, 0) onto (–3, 1), and maps (0, 1) onto (2, –1), what is the transformation matrix?

Answer

(1, 0) is mapped onto (–3, 1), so the first column becomes $\begin{pmatrix} -3 & \cdots \\ 1 & \cdots \end{pmatrix}$

(0, 1) is mapped onto (2, –1), so the second column becomes $\begin{pmatrix} \cdots & 2 \\ \cdots & -1 \end{pmatrix}$

Hence the transformation matrix is $\begin{pmatrix} -3 & 2 \\ 1 & -1 \end{pmatrix}$.

SUB-TOPIC 3 Reflection

A reflection is a transformation in which any two corresponding points of the object and the image are the same distance from, and at right angles to, a straight line, called the mirror line.

Reflection is usually denoted by the matrix M.

Fig. 5.6 shows a flag with vertices QRSPT. The flag has been reflected in the mirror line M whose equation is $y = x$.

The shape Q'R'S'P'T' is the image of QRSPT. We can find the image of Q' by drawing a perpendicular line from Q to the mirror line, and then extending the line to the other side of the mirror line by an equal length.

We can do the same for the other points.

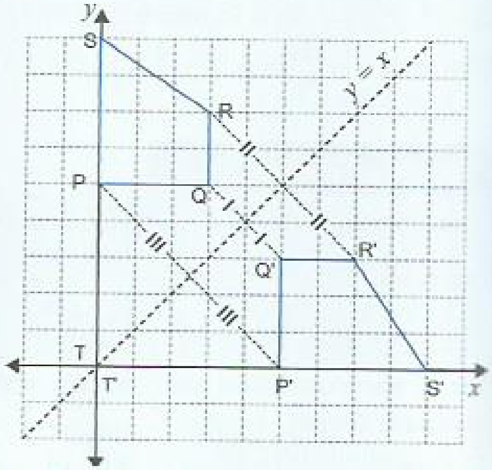


Figure 5.6

Note

In a reflection the corresponding points of the object Q and its image Q' are the same distance from the mirror line. Therefore the line $y = x$ is a perpendicular bisector of QQ'.

We write M(Q) = Q'. This means the point Q is mapped onto Q' under reflection M. Points, such as T (in Fig. 5.6), which lie on the mirror line are invariant, which means that this point does not move, and so M(T) = T'.

Worked example 4

1 If P(1, 0) and Q(0, 1), find P' and Q' and hence the matrices of the reflections in the following lines:
   - a) x-axis
   - b) $y = x$
   - c) $y = -x$

2 a) Draw the quadrilateral PQRS at P(−4, 2), Q(−1, 2), R(−1, 4) and S(−2, 5).
   - b) Draw the line $x = 1$.
   - c) Reflect the quadrilateral PQRS in $x = 1$.

3 Find the coordinates of the image of ΔPQR with vertices P(1, 1), Q(1, 3) and R(2, 1) under a reflection in the line $y = -x$.

Worked example 4 (continued)

Answers

1 a) The image P is invariant as it lies on the x-axis (mirror line of reflection).
The point Q(0, 1) is mapped onto Q' = (0, –1).
So the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is a reflection in the x-axis.

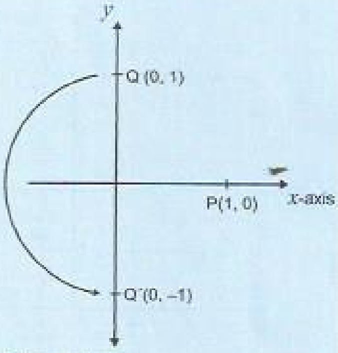


Figure 5.7

b) The point P is mapped onto Q(0, 1).
The point Q is mapped onto P(1, 0).
Hence the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is a reflection in the line $y = x$.

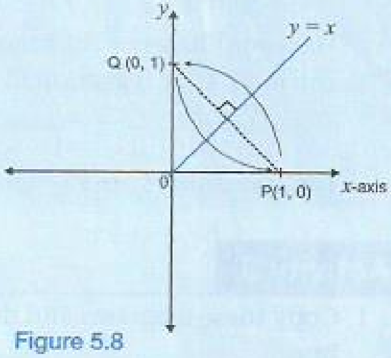


Figure 5.8

c) The point P is mapped onto P'(0, –1).
The point Q is mapped onto Q'(–1, 0).
Hence the matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ is a reflection in the line $y = -x$.

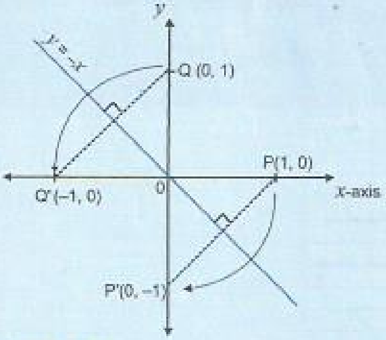


Figure 5.9

Ct

Worked example 3 (continued)

2

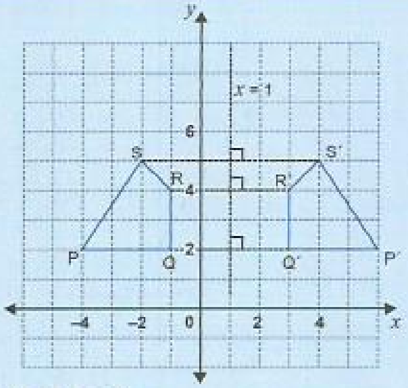


Figure 5.10

The coordinates of the image are P'(6, 2), Q'(3, 2), R'(3, 4) and S'(4, 5)

3 The matrix for reflection in the line $y = -x$ is $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$:

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -3 & -1 \\ -1 & -1 & -2 \end{pmatrix}$$

The coordinates are P'(−1, −1), Q'(−3, −1) and R'(−1, −2).

Activity 3

1 Copy these diagrams and draw each image after reflection in the broken line.

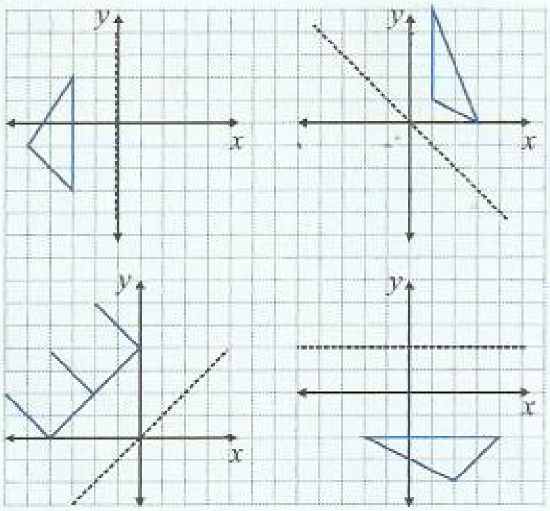


Figure 5.11

2 ΔABC has vertices A(−5, 6); B(−5, 1); C(−2, 1).
Find the coordinates of the image of ΔABC if the triangle is reflected in the y-axis.

Activity 3

3 Draw ΔP
reflection
a) $y = 1$.
b) $y = x$.
c) $y = -x$
d) $y = 0$.
4 a) Copy
b) Find t
line fo
the fo
line in
i) Δ1
ii) Δ2
iii) Δ1
iv) Δ5
5 Draw Δ1 w
and (6, 4)
a) Reflect
b) Reflect
c) Reflect
d) Reflect
e) Write d

Summary

A reflection is a t
creating a mirror
mirror line.

Activity 3 (continued)

3 Draw ΔPQR with vertices P(0, –4), Q(3, –2) and R(3, 0). Draw its image after reflection in each of the following lines:

- a) $y = 1$. Label it Δ1.
- b) $y = x$. Label it Δ2.
- c) $y = -x$. Label it Δ3.
- d) $y = 0$. Label it Δ4.

4 a) Copy Fig. 5.12.

- b) Find the equation of the mirror line for the reflection of each of the following. Draw the mirror line in each case.
  - i) Δ1 → Δ2
  - ii) Δ2 → Δ4
  - iii) Δ1 → Δ3
  - iv) Δ5 → Δ4

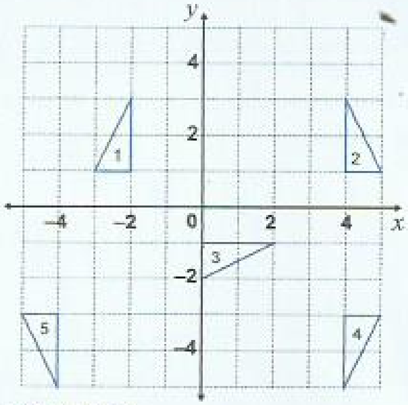


Figure 5.12

5 Draw Δ1 with vertices (2, 4), (2, 6) and (6, 4)

- a) Reflect Δ1 in the line $y = x$ onto Δ2.
- b) Reflect Δ2 in the x-axis onto Δ3.
- c) Reflect Δ3 in the line $y = -x$ onto Δ4.
- d) Reflect Δ4 in the line $x + y = 2$ onto Δ5.
- e) Write down the coordinates of Δ5.

Summary

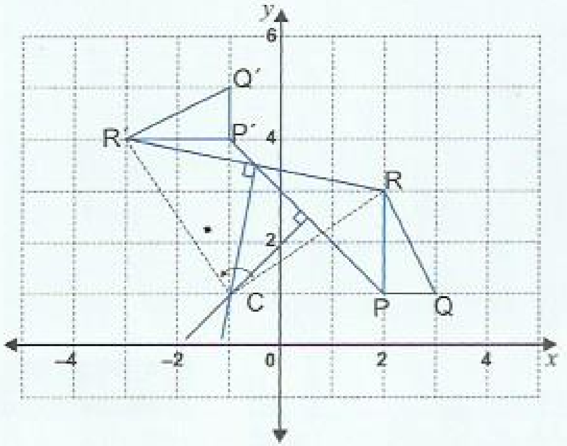
A reflection is a transformation in which a geometric figure is reflected across a line, creating a mirror image. To fully describe a reflection we give the equation of the mirror line.

SUB-TOPIC 4 Rotation

A rotation is a transformation in which an object is turned about a fixed point called the centre of rotation.

Rotation is usually denoted by the matrix R.

To fully describe a rotation, you need to give the centre, angle and direction of the rotation.



Note

A clockwise direction means that the rotation is negative: an anticlockwise direction means it's positive.

Figure 5.13

Fig. 5.13 shows ΔP'Q'R' after a rotation of ΔPQR.

How to find the centre of rotation

Step 1: Join a pair of corresponding points such as P and P'.
Step 2: Construct a perpendicular bisector of the line PP'.
Step 3: Repeat for another pair of corresponding points such as R and R'.

The centre of rotation is at the point of intersection of the two perpendicular bisectors. In Fig. 5.13 the centre of rotation is at C (−1, 1).

How to find the angle and direction of rotation

Step 1: Join a point on the object to the centre of rotation, e.g. RC.
Step 2: Join its image to the centre of rotation, e.g. R'C.
Step 3: Measure the angle formed between RC and R'C.

In Fig. 5.13, the angle of rotation is 90° anticlockwise.

Worked exa

1 Find the ma
a) 90° anti
Find the im
2 ΔABC has v
a clockwise
3 ΔPQR has ve
an anticlock

Answers

a) The point P
The point Q
Hence the m
of 90° anticl
b) The point P
The point Q
Hence the m
of 180° abou

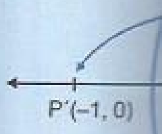


Figure 5.15

c) The point P is
The point Q i
Hence the ma
of 270° anticl
This is equiva
90° clockwise.

Worked example 5

1 Find the matrices for rotation about the origin O(0, 0) for the following:
a) 90° anticlockwise b) 180° c) 270° anticlockwise
Find the images of P(1, 0) and Q(0, 1) in each case.

2 ΔABC has vertices A(3, 2), B(5, 2) and C(5, 4). Find the image of ΔABC under a clockwise rotation of 90° about O.

3 ΔPQR has vertices P(4, 2), Q(7, 2) and R(7, 0). Draw the image ΔPQR under an anticlockwise rotation of 90° with centre (1, 1).

Answers

a) The point P is mapped onto Q(0, 1).
The point Q is mapped onto Q'(−1, 0).
Hence the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is a rotation ?
of 90° anticlockwise about the origin.

b) The point P is mapped onto P'(−1, 0).
The point Q is mapped onto Q'(0, −1).
Hence the matrix is $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is a rotation of 180° about the origin.

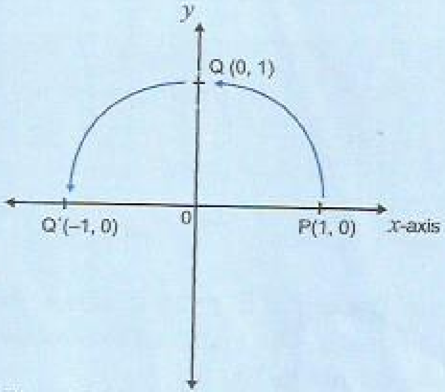


Figure 5.14

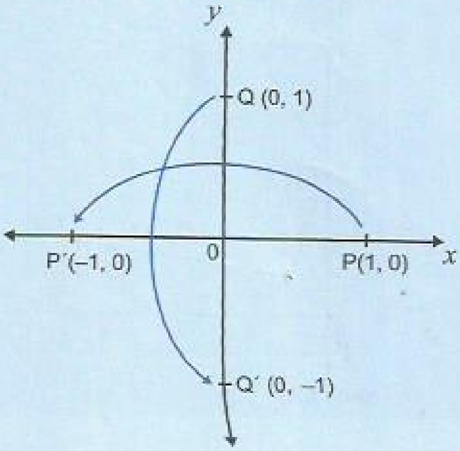


Figure 5.15

c) The point P is mapped onto P'(0, −1)
The point Q is mapped onto P(1, 0).
Hence the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is a rotation of 270° anticlockwise about the origin.
This is equivalent to a rotation of 90° clockwise.

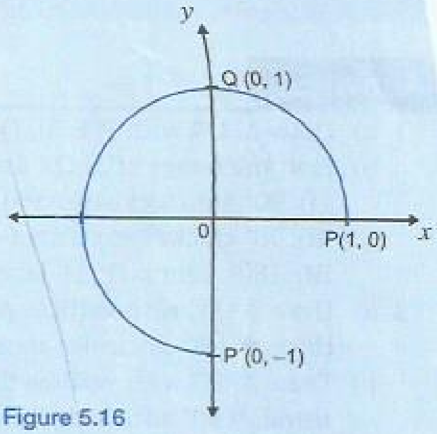


Figure 5.16

Worked example 5 (continued)

2 Matrix for 90° clockwise rotation

$$= \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

Multiply the three column vectors for A, B and C as a 2 × 3 matrix.

$$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{array}{c}\begin{array}{ccc}A & B & C\end{array}\\ \begin{pmatrix}3 & 5 & 5\\ 2 & 2 & 4\end{pmatrix}\end{array} = \begin{array}{c}\begin{array}{ccc}A' & B' & C'\end{array}\\ \begin{pmatrix}2 & 2 & 4\\ -3 & -5 & -5\end{pmatrix}\end{array}$$

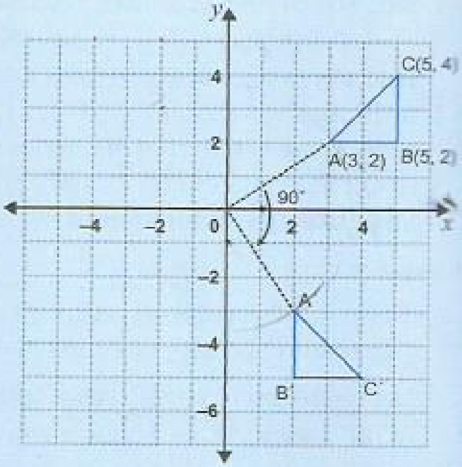


Figure 5.17

3 The centre of rotation is not the origin. The matrix of this rotation cannot be found.

Join P to the centre C (1, 1). Then measure 90° anticlockwise and with the same distance as CP, measure CP' and mark P'.

Do the same for Q and R as shown in Fig. 5.18.

ΔP'Q'R' has vertices P'(0, 4), Q'(0, 7) and R'(2, 7)

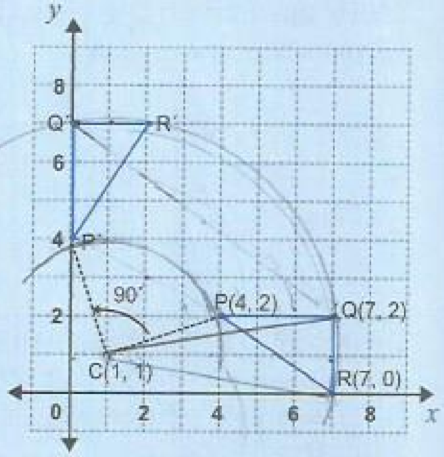


Figure 5.18

Activity 4

1 a) Draw ΔPQR with P(1, 3), Q(3, 6) and R (6, 2)

b) Find the image of ΔPQR under the following rotations:

- i) 90° anticlockwise, centre (0, 0), label the image $P_1Q_1R_1$
- ii) 90° clockwise, centre (–2, 2); label the image $P_2Q_2R_2$
- iii) 180°, centre (1, 1), label the image $P_3Q_3R_3$

2 a) Draw ΔABC with vertices A(4, 3), B(7, 3) and C(7, 1). Rotate ΔABC through 90° clockwise about (0, 0), mark A'B'C'.

b) Draw ΔEFG with vertices E(–6, 7), F(–6, 5) and G(–3, 5), rotate ΔEFG through 90° anticlockwise about (0, 0) mark E'F'G'.

Activity 4 (

3 ΔABC with
vertices F(–
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b) Find th
c) Find th
4 a) Draw a
b) A trans
B'(–3, 4

Summary

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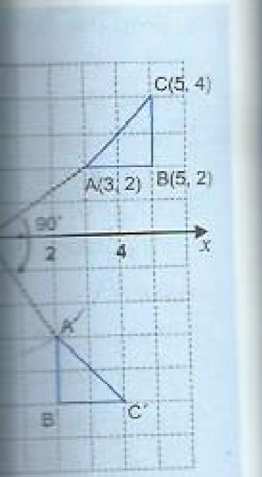

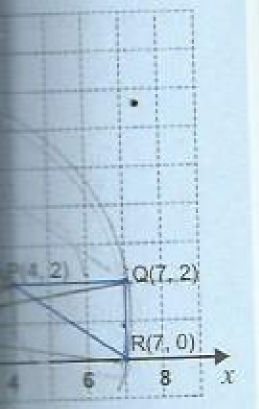

s:

Q_1R_1

R_2

Rotate ΔABC

5), rotate ΔEFG

Activity 4 (continued)

3 ΔABC with vertices A(6, 12), B(8, 4) and C(2, 2) is mapped onto ΔFGH with vertices F(–12, 6), G(–4, 8) and H(–2, 2) by a rotation.
- a) Draw and label ΔABC and ΔFGH.
- b) Find the centre of rotation by *construction*.
- c) Find the angle of rotation and its direction.

4 a) Draw and label ΔABC whose vertices are A(0, 1), B(2, 1) and C(3, 3).
- b) A transformation maps ΔABC onto ΔA'B'C' with vertices A'(–3, 2), B'(–3, 4) and C'(–5, 5). Describe this transformation fully.

Summary

A rotation is a transformation where an object is turned about a fixed point. To fully describe a rotation we give the centre of rotation and the angle and direction of the rotation.

SUB-TOPIC 5 Enlargement

An enlargement is a transformation by which an object changes in size by either being magnified (made larger) or diminished (made smaller).

Enlargement is usually denoted by the matrix E.

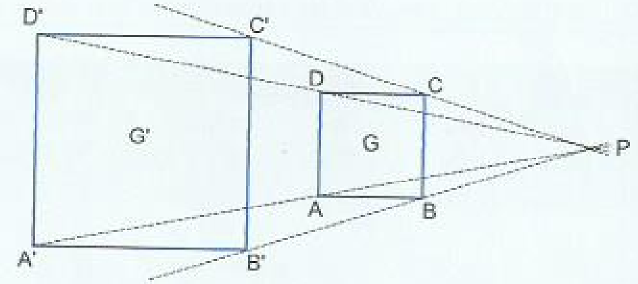

Figure 5.19

In Fig. 5.19, square G has been transformed into G' by an enlargement.

- G and G' are not isometric (congruent). They are similar to each other and the corresponding sides are in the same ratio.
- P is the centre of enlargement. The centre of enlargement is invariant (fixed).
- The quantity by which an object is enlarged is called a scale factor. The ratio of the lengths of any pair of corresponding sides gives the scale factor.

$$\text{Scale factor} = \frac{\text{Image length}}{\text{Object length}}$$

If k is the scale factor of a transformation E, then:

- $PA' = kPA$
- If $k = 1$, the image is the same as the object.
- If $k > 1$, the image is larger than the object.
- If k is negative, the image is turned around. A negative scale indicates that the image is on the other side of the centre of enlargement from the object.
- The area of an image $P'Q'R' = k^2 \times$ (area of object PQR).
  Example: If $k = \frac{1}{2}$, then the area of $P'Q'R' = \frac{1}{4}$ (area of PQR).

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P

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the object.

Finding the centre of enlargement and the scale factor

In Fig. 5.20 we are given an object G and its image G' and we need to find the centre of enlargement and scale factor.

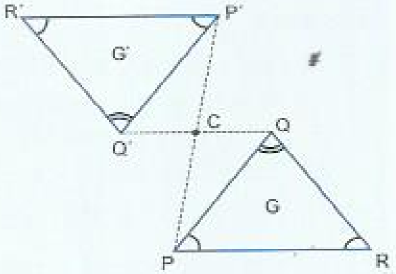

Figure 5.20

Join the corresponding vertices, that is, a point and its image, such as PP', QQ' etc. The point of intersection of these lines C is the centre of enlargement.

Worked example 6

Find the matrix for enlargement with centre O(0, 0) and scale factor k.

Answer

The point P(1, 0) is mapped onto P' $(k, 0)$
The point Q(0, 1) is mapped onto Q' $(0, k)$.

Hence the matrix is $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

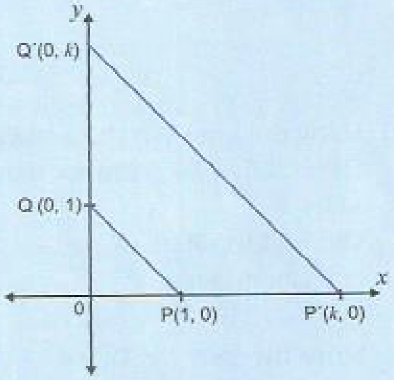

Figure 5.21

The matrix $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ represents an enlargement E with scale factor k and centre O(0, 0)

Worked example 7

1 ΔABC with vertices A(–4, 3), B(–2, 3) and C (–4, 4) is mapped on to ΔA'B'C' with vertices A'(5; 0), B'(1, 0) and C'(5, –2) by an enlargement matrix E. By drawing the triangles on the graph, find
   a) the coordinates of the centre of enlargement
   b) the scale factor of E.

2 ΔPQR has vertices P(3, 0); Q(3, 2) and R(2, 2). Draw the image of ΔPQR under an enlargement scale factor of 2 with O(0, 0) as the centre of enlargement.

3 ΔABC with vertices A(1, 1), B(2, 1) and C(1, 3) is enlarged by a scale factor of –3 with the origin as centre. Find the coordinates of ΔA'B'C'.

Worked example 7 (continued)

Answers

1 a) The lines AA' and BB' intersect at P, the centre of enlargement. P has coordinates (–1, 2).

b) Scale factor $= \frac{PA'}{PA}$

$= \frac{A'C'}{AC}$

$= \frac{-2}{1}$

$= -2$

or $\frac{A'B'}{AB} = \frac{-4}{2}$

$= -2$

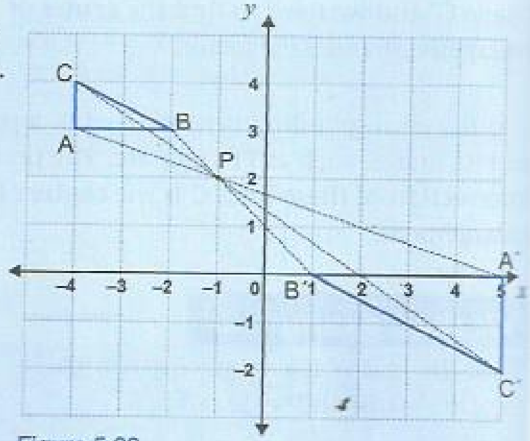

Figure 5.22

2 Draw the lines OP, OQ and OR.

OP' = 2OP = 2 × 3 units = 6 units. Mark P'

OQ' = 2OQ. Mark Q'.

OR' = 2OR. Mark R'.

or

Since the centre is O, we can use the matrix for enlargement $\begin{pmatrix}k&0\\0&k\end{pmatrix}$.

Scale factor = 2 so E $= \begin{pmatrix}2&0\\0&2\end{pmatrix}$

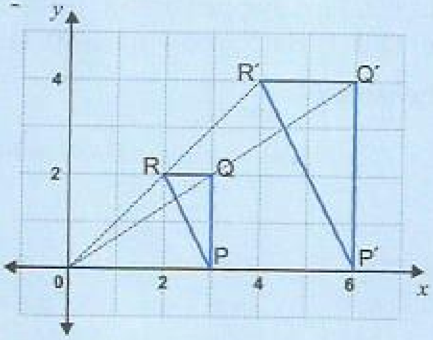

Figure 5.23

$$\begin{pmatrix}2&0\\0&2\end{pmatrix}\begin{array}{c}\begin{array}{ccc}P&Q&R\end{array}\\ \begin{pmatrix}3&3&2\\0&2&2\end{pmatrix}\end{array}=\begin{array}{c}\begin{array}{ccc}P'&Q'&R'\end{array}\\ \begin{pmatrix}6&6&4\\0&4&4\end{pmatrix}\end{array}$$

The coordinates are P'(6, 0), Q'(6, 4) and R'(4, 4)

3 Since the centre of enlargement is 0, we can use the matrix.

Scale factor = –3, so E $= \begin{pmatrix}-3&0\\0&-3\end{pmatrix}\begin{array}{c}\begin{array}{ccc}A&B&C\end{array}\\ \begin{pmatrix}1&2&1\\1&1&3\end{pmatrix}\end{array}=\begin{array}{c}\begin{array}{ccc}A'&B'&C'\end{array}\\ \begin{pmatrix}-3&-6&-3\\-3&-3&-9\end{pmatrix}\end{array}$

The coordinates are A'(–3, –3), B'(–6, –3) and C'(–3, –9).

Activity 5

1 Fig. 5.24 sh
of triangle T
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factor of the
enlargemen
a) T_1 onto
b) T_2 onto
c) T_1 onto
d) R_1 onto

2 Copy Figs. 5
of the enlarg

Figure 5.25

3 ΔPQR has ver
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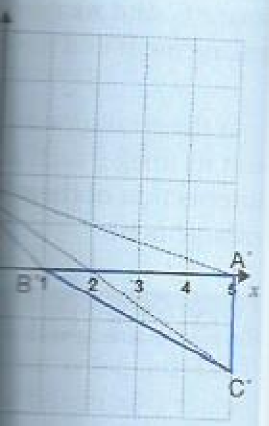
scale and dra



Activity 5

1 Fig. 5.24 shows some enlargements of triangle T and rectangle R. In each case, the origin is the centre of enlargement. State the scale factor of the following enlargements:

a) T_1 onto T_2
b) T_2 onto T_3
c) T_1 onto T_3
d) R_1 onto R_2

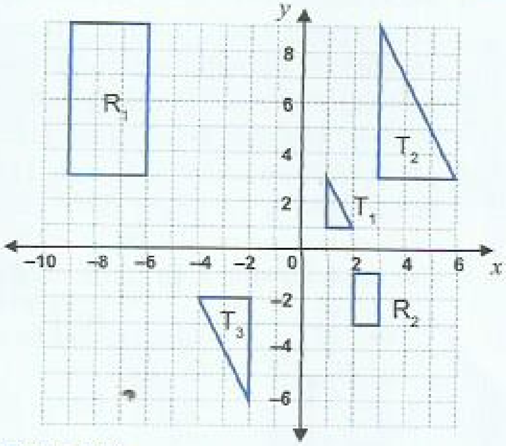


Figure 5.24

2 Copy Figs. 5.25a to Fig. 5.25d and draw enlargements using O as the centre of the enlargement and the scale factor given in each case.

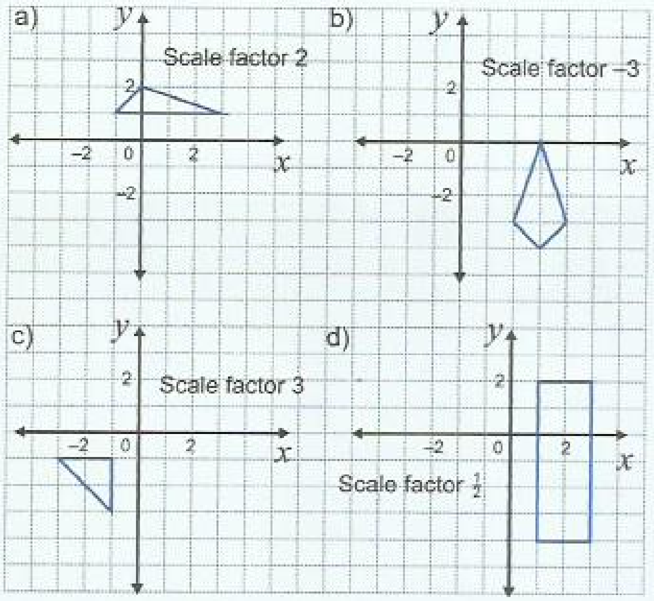


Figure 5.25

3 ΔPQR has vertices P(4, –1), Q(5, –2) and R(4, –4). ΔPQR is enlarged by a scale factor of –2 with (3, –2) as the centre of enlargement. Choose a suitable scale and draw ΔPQR with its image ΔP'Q'R'.

Activity 5 (continued)

4 ΔABC has vertices A(–3, 1), B(–3, 4) and C(–1, 4). It is enlarged to ΔA'B'C' with vertices A'(–1, 3), B'(–1, –3) and C(–5, –3).

a) Choose a suitable scale and draw ΔABC and its image.

b) Find the centre of enlargement.

c) State the scale factor of the enlargement.

5 ΔPQR with vertices P(3, 1), Q(6, 1) and R(6, –1) is enlarged by scale factor $-\frac{1}{2}$ with the origin as the centre of enlargement. Find the coordinates of its enlargement ΔP'Q'R'.

6 Use the following matrices to enlarge ΔABC in Fig. 5.26.

a) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ b) $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$

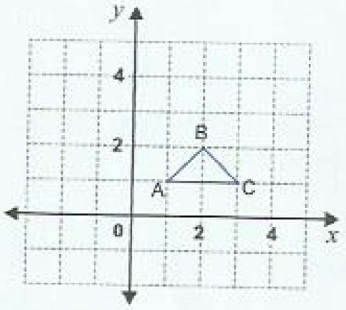


Figure 5.26

Summary

An enlargement is a transformation in which an object changes it size by being made larger or smaller. To fully describe an enlargement we need to give the centre of enlargement and a scale factor.

SUB-TOPIC 6

A stretch is
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Stretch is us

In Fig. 5.27 a
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SUB-TOPIC 6 Stretch

A stretch is a transformation in which all points along a given line remain fixed while other points are shifted parallel to the line by a distance proportional to their perpendicular distance from the line.

Stretch is usually denoted by the matrix S.

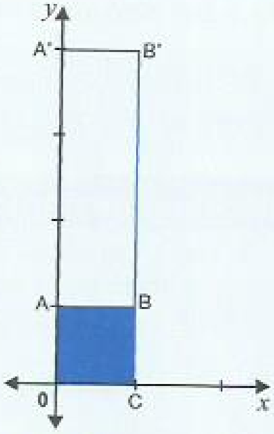

Figure 5.27

In Fig. 5.27 a unit square OABC has been transformed into OA'B'C by a stretch. Note the following:

- OC is the invariant line. All the other points on the object, except those on OC, have moved in a direction perpendicular to the invariant line.
- The unit square OABC has been stretched in the direction of the *y*-axis. AB is mapped onto A'B'.
- Stretch factor = $\frac{\text{distance of image from invariant line}}{\text{distance of object from invariant line}}$

$= \frac{OA'}{OA} = \frac{CB'}{CB} = \frac{4}{1} = 4$

To describe a stretch in full you need to do the following:

- Identify the invariant line.
- State the direction of the stretch.
- Give the scale factor for the stretch.

Worked example 8

Find the stretch matrix S such that the *y*-axis is invariant and the point A(1, 2) is mapped onto A'(3, 2).
Find the images of P(1, 0) and Q (0, 1) under S.

Answer

The point A(1, 2) moves 2 units to A'(3, 2).
So P(1, 0) also moves 2 units to P'(3, 0).
OQ is invariant, Q remains at (0, 1).
Hence the matrix $S = \begin{pmatrix}3 & 0\\0 & 1\end{pmatrix}$.

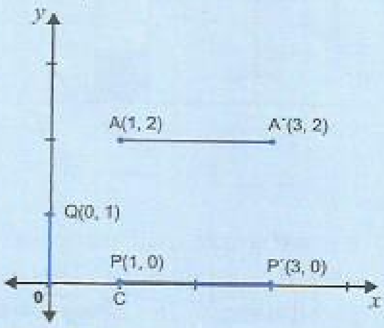

Figure 5.28

The matrix $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$ represents a one-way stretch S with the y-axis invariant and a scale factor k.

The matrix $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$ represents a one-way stretch S with the x-axis invariant and scale factor k.

Two one-way stretches may be combined to give a two-way stretch.

An example of a two-way stretch is $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$. Here the stretch has a factor of 3 in the x-direction and 4 in the y-direction.

The matrix $\begin{pmatrix} h & 0 \\ 0 & k \end{pmatrix}$ represents a two-way stretch S of factor h in the x-direction and k in the y-direction.

Worked example 9

1 Fig. 5.29a, Fig. 5.29b and Fig. 5.29c each shows a square and its image under a stretch. Describe each stretch fully, giving the stretch factor, its direction and the equation of the invariant line.

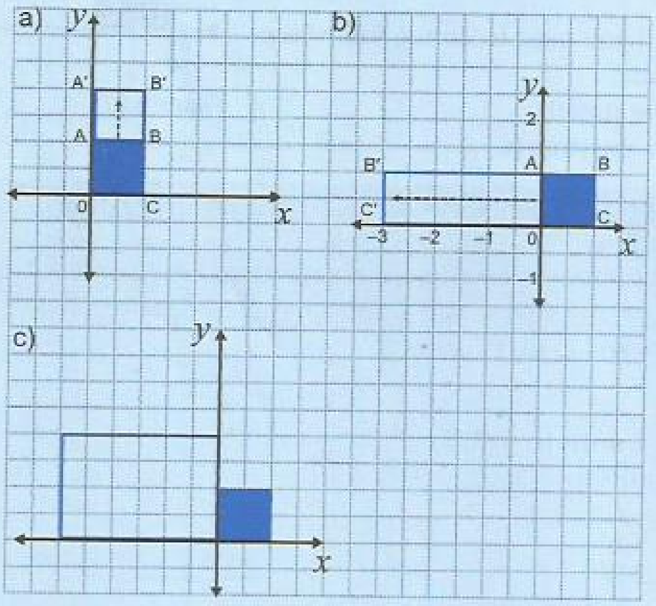

Figure 5.29

2 T is a triangle with vertices A(2, 2), B(4, 2) and C(2, 6). ΔT is given a one-way stretch S of factor 2 in the x-direction, with the y-axis invariant. Find the coordinates of the image of ΔT.

Worked exa

3 a) Find the
S(0, 1) ur
b) Describe

Answers

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the y-axis
b) Stretch fa
This trans
x-axis wit
c) The two c
stretch.
The trans
and −3 in

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x-coordinate
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B'(2 ×
C'(2 ×

3 a) $\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
The coord
S'(0, 5).
b) The trans
and 5 in t
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xis invariant and

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Worked example 8 (continued)

3 a) Find the image of the square PQRS with vertices P(0, 0), Q(1, 0), R(1, 1), S(0, 1) under the transformation represented by the matrix $H = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$.

b) Describe the transformation H in full.

Answers

1 a) Stretch factor $= \frac{OA'}{OA} = \frac{CB'}{CB} = 2$

The transformation is a one-way stretch of factor 2 in the direction of the y-axis with the x-axis invariant.

b) Stretch factor $\frac{AB'}{AB} = -3$

This transformation is a one-way stretch of factor -3 in the direction of x-axis with the y-axis invariant.

c) The two one-way stretches in (a) and (b) are combined to give a two-way stretch.

The transformation is a two-way stretch of factor 2 in the y-axis direction and -3 in the x-axis direction.

2 A stretch of factor 2 in the x-axis direction has the effect of multiplying each x-coordinate of each vertex of ΔT by 2. See Fig. 5.30.

$S(T) = A'(2 \times 2, 2) = (4, 2)$

$B'(2 \times 4, 2) = (8, 2)$

$C'(2 \times 2, 6) = (4, 6)$

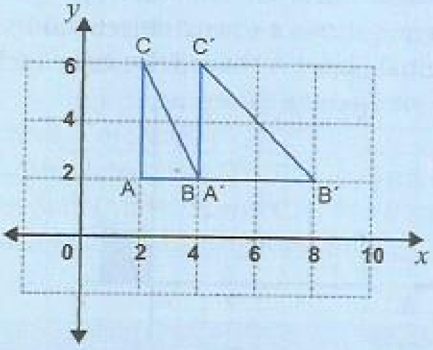

Figure 5.30

3 a) $\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 5 & 5 \end{pmatrix}$

The coordinates of the image of the square are P'(0, 0), Q'(2, 0), R'(2, 5), S'(0, 5).

b) The transformation is a two-way stretch of factor 2 in the x-direction and 5 in the y-direction.

Notice that the origin is always mapped onto itself when multiplied by a 2 × 2 matrix.

Activity 6

1 In Fig. 5.31 each diagram shows a square object and its image after a one-way stretch. The image is shaded in each case.

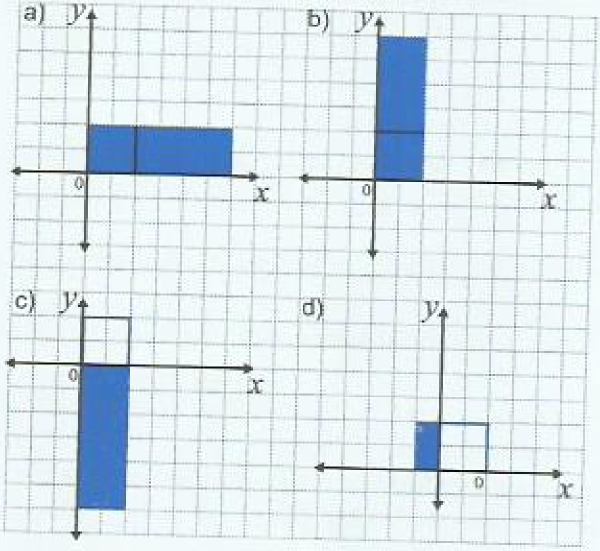

Figure 5.31

Describe fully each stretch, giving the stretch factor, the direction and the equation of the invariant line.

2 In Fig. 5.32 each diagram shows a square object and its image after a two-way stretch. The original object is shaded. Describe each stretch fully.

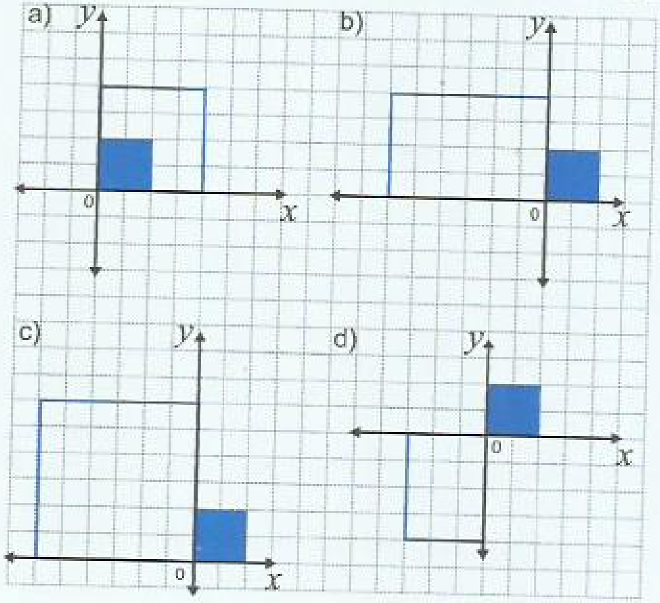

Figure 5.32

Activity 6 (c

3 The square
x-direction
a) Find th
b) What k

4 ΔABC with
factor −2 in
image ΔA'B'C

5 ΔPQR with
vertices P'(1,
a) the matri
b) the scale

6 Draw rectang
stretch of fac
a) Find the
b) What kin

7 ΔABC with v
matrix $\begin{pmatrix}3 & 0\\0 & 2\end{pmatrix}$

Summary

A stretch is a transf
way stretch) or by t
To fully describe a s
stretch and give the

Activity 6 (continued)

3 The square ABCD in Fig. 5.33 is given a one-way stretch of factor 2 in the x-direction with the y-axis invariant.

  a) Find the coordinates of the image A'B'C'D'.

  b) What kind of shape is A'B'C'D'?

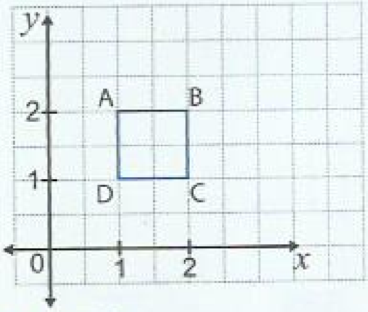

Figure 5.33

4 ΔABC with vertices A(2, 1), B(5, 1) and C(3, 5) is given a one-way stretch of factor –2 in the x-direction with the y-axis invariant. Draw ΔABC and its image ΔA'B'C' after the stretch.

5 ΔPQR with vertices P(1, 1), Q(2, 1) and R(2, 2) is mapped onto ΔP'Q'R' with vertices P'(1, –3), Q'(2, –3) and R'(2, –6) by a stretch. Find:

  a) the matrix which represents this transformation

  b) the scale factor of the stretch.

6 Draw rectangle PQRS with vertices at P(0, 0), Q(0, 2), R(3, 2), S(3, 0). S is a stretch of factor 3 in the x-direction with the y-axis invariant.

  a) Find the coordinates of image P'Q'R'S'

  b) What kind of quadrilateral is P'Q'R'S'?

7 ΔABC with vertices A(1, 1), B(2, 3) and C(4, 2)is transformed by using the matrix $\begin{pmatrix}3 & 0\\0 & 2\end{pmatrix}$. Describe the transformation fully.

Summary

A stretch is a transformation characterised by an invariant line and a scale factor (one-way stretch) or by two invariant lines and two corresponding factors (two-way stretches). To fully describe a stretch, we identify the invariant line(s), state the direction(s) of the stretch and give the scale factor(s) of the stretch.

Shear

A shear is a transformation in which all points along a given line remain fixed, while other points are shifted parallel to the line by a distance proportional to their perpendicular distance from the line.

Shear is usually denoted by the matrix H.

The square OABC is transformed into OA'B'C by a shear.

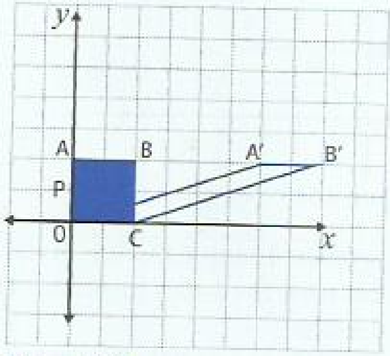

Figure 5.34

From the shear in Fig. 5.34, note:

- OC is the invariant or fixed line.
- All the other points on the square, except OC, have moved parallel to the invariant line, the x-axis.
- The distance moved by any point depends on its distance from the invariant line. The point A on the top of the square moves twice as far as the point P in the middle. The points on both sides of the invariant line move by an amount proportional to their distances from the line.
- Shear factor = $\frac{\text{distance moved by a point}}{\text{distance of that point from the invariant line}} = \frac{AA'}{OA} = \frac{BB'}{CB} = \frac{3}{1} = 3$

To describe a shear in full you need to:

- Identify the invariant line.
- Give the shear factor.
- State the direction of the shear.

The matrix $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ represents a shear with the x-axis invariant and shear factor k.

The matrix $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$ represents a shear with the y-axis invariant and shear factor k.

Worked e

1 Describe
equation

Figure 5.35

2 Find the c
C(1, −2) a
a) a shear
b) a shear
3 H is a tran
a) Find th
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Answers

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shear fa
2 a) Use the
Shear fa
A(4, 0) is
For B(1,

ven line remain fixed,
tance proportional to

ed parallel to the

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s far as the point P in
e move by an amount

$\frac{AA'}{OA} = \frac{BB'}{CB} = \frac{3}{1} = 3$

iant and shear

iant and shear

Worked example 10

1 Describe each shear fully, giving the shear factor, the direction and the equation of the invariant line.

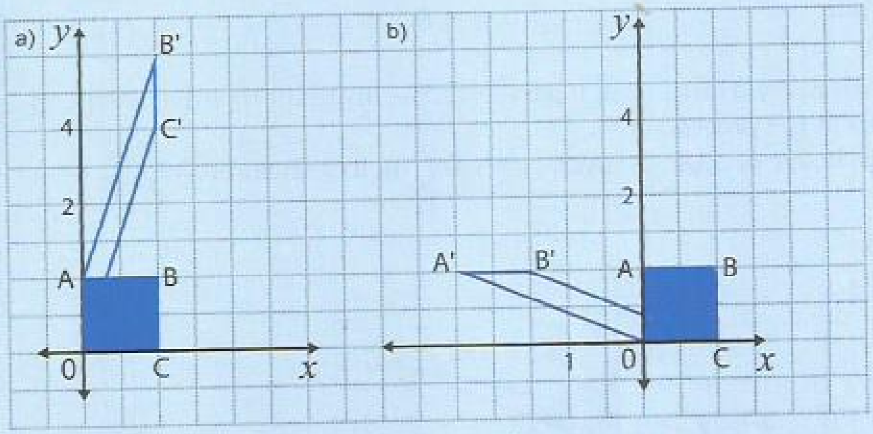


Figure 5.35

2 Find the coordinates of the image of ΔABC with vertices A(4; 0), B(1; 2) and C(1, –2) after the following:

a) a shear of factor 4 with the x-axis invariant

b) a shear of factor –1 with the y-axis invariant.

3 H is a transformation represented by the matrix $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

a) Find the image of ΔABC with vertices A(–2, 3), B(–2, 5) and C(–5, 5).

b) Describe the transformation H in full.

Answers

1 a) Shear factor $= \frac{BB'}{AB} = \frac{3}{1} = 3$

This is a shear parallel to the y-axis with $x = 0$ as the invariant line and shear factor of 3.

b) Shear factor $= \frac{AA'}{OA} = \frac{-2\frac{1}{2}}{1} = -2\frac{1}{2}$

The shear factor is negative, since the shear is in the negative direction of the x-axis.

The shear is parallel to the x-axis with $y = 0$ as the invariant line and shear factor of $-2\frac{1}{2}$.

2 a) Use the shear factor to find the images of A, B and C.

$$\text{Shear factor} = \frac{\text{distance moved by point}}{\text{distance of that point from the invariant line}}$$

A(4, 0) is on the invariant line; it cannot move.

For B(1, 2), let a be the unknown distance moved by the object.

Worked example 10 (continued)

$$\text{Distance of B from invariant line} \longleftarrow \frac{a}{2} = 4 \longrightarrow \text{The given shear factor}$$

$$a = 8$$

Then move 8 units from B parallel to the invariant to arrive at B'(9, 2).
Repeat the process for C.
The image of ΔABC is ΔAB'C' with A(4, 0), B'(9, 2) and C'(–7, –2).

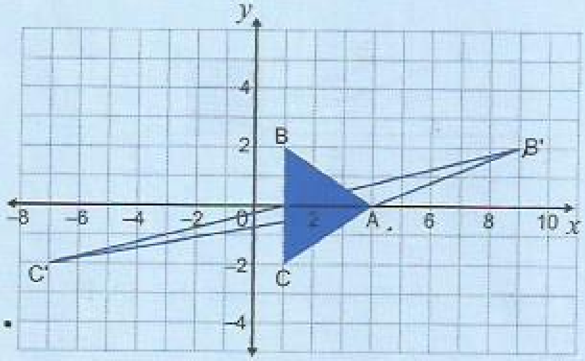


Figure 5.36a

b) The images of A, B and C after a shear factor of –1 with the *y*-axis invariant are A'(4, –4), B'(1, 1) and C'(1, –3).
Fig. 5.36b shows the image ΔA'B'C'.

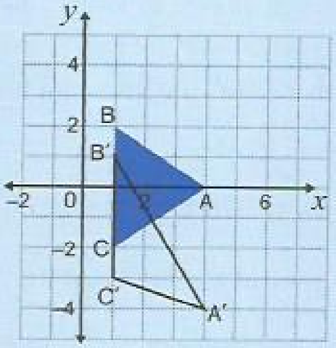


Figure 5.36b

3 The transformation matrix H is $\begin{pmatrix}1 & 3\\0 & 1\end{pmatrix}$

a) The coordinates of the image of ΔABC are A'(7, 3), B'(13, 5) and C'(10, 5)

b) H is a shear with *x*-axis invariant and shear factor 3.

Activity 7

In Fig. 5.37, e
shear.

1

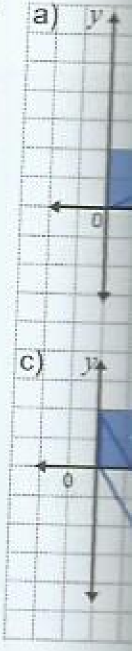


Figure 5.37

Describe ful
equation of

2 Square ABC
with the *x*-a

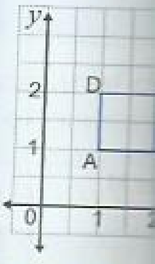


Figure 5.38

a) Find the c

b) What kin

given
ar factor

at B'(9, 2).

7, -2).

the y-axis

5) and C'(10, 5)

Activity 7

In Fig. 5.37, each diagram shows a rectangular object and its image under a shear.

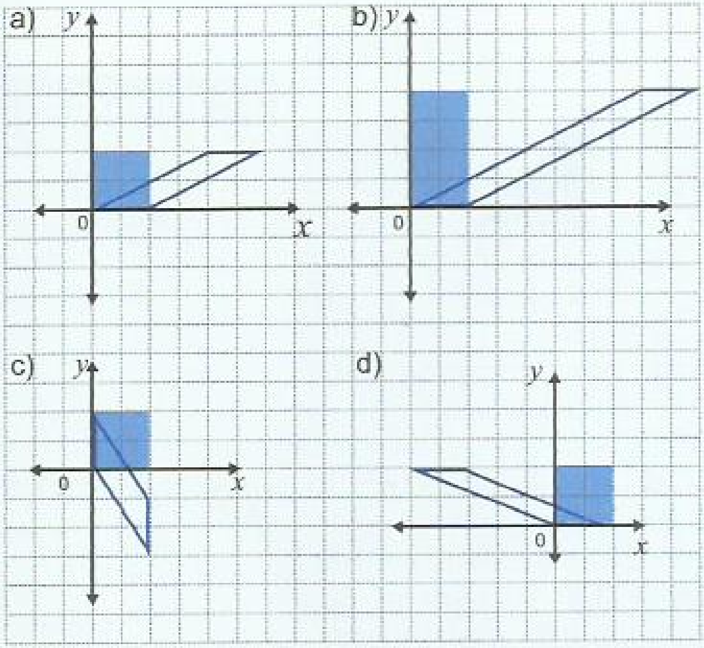
1 

Figure 5.37

Describe fully each shear, giving the shear factor, the direction and the equation of the invariant line.

2 Square ABCD in Fig. 5.38 is given a shear H of factor 2 in the x-direction with the x-axis invariant.

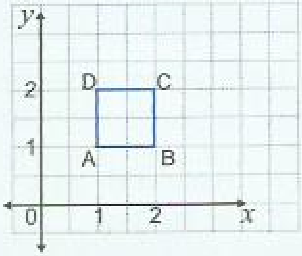


Figure 5.38

Note

Another way of indicating the image under translation is showing matrix multiplication. e.g. H(ABCD).

a) Find the coordinates of H(ABCD).

b) What kind of shape is H(ABCD)?

Activity 7 (continued)

3 The single transformation H maps ΔA onto ΔB. Describe fully the transformation H.

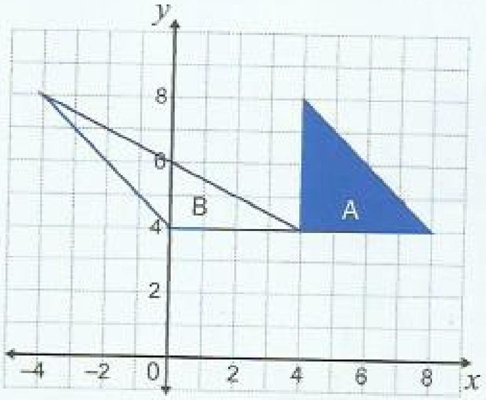


Figure 5.39

4 Find the coordinates of the image of ΔABC with vertices A(−1, 0), B(1, 1) and C(2, −1) after the following:

a) a shear of factor 2 with the x-axis invariant

b) a shear of factor −3 with the x-axis invariant

c) a shear of factor 1 with the y-axis invariant.

5 H is a shear of factor $-1\frac{1}{2}$ in the x-direction with the x-axis invariant. P is the point (−3, 5) and Q is (4, 3).

Find the coordinates of:

a) i) H(P) ii) H(Q)

b) State the length of H(PQ)

6 The matrix $\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$ represents a transformation H.

a) Find the image of (3, −4) under H.

b) Find the image of (−4, 3) under H

c) Describe the transformation H in full.

7 A shear H is represented by the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

a) Calculate the coordinates of the image of Q(−2, −3) under H.

b) Calculate the coordinates of the point which will be mapped onto (6, 2) by H.

c) Write down the equation of the invariant line.

Summary

A shear is a transformation in which all points along a given line remain fixed, while other points are shifted parallel to the line by a distance proportional to their perpendicular distance from the line. To fully describe a shear transformation, we give the shear factor, the direction and the equation of the invariant line.

Transformations
order.

Worked exa

1 ΔT has vertic
M is a reflect
with the line
Sketch the tr
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Answer

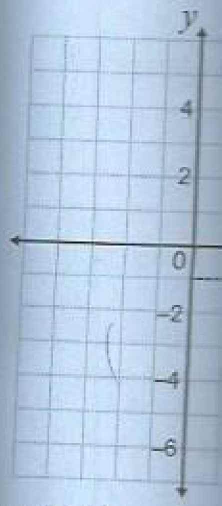

Figure 5.40

The order of co

g. 5.40 shows ΔT a
e final image whe
ritten as HM(T).

Note

HM(T) means first app
H (shear) to that.
The order in which you

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A(−1, 0), B(1, 1) and

is invariant.

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n fixed, while other
r perpendicular
e the shear factor,

SUB-TOPIC 8 Combined transformations

Transformations may be combined, with each transformation applied strictly in order.

Worked example 11

1 ΔT has vertices A(2, −2), B(2, −6) and C(4, −6).
M is a reflection in the line $y = -1$. H is a shear of factor 2 in the x-direction with the line $y = 0$ invariant.
Sketch the transformations and show the final image if ΔT is first reflected by M and then sheared by H.

Answer

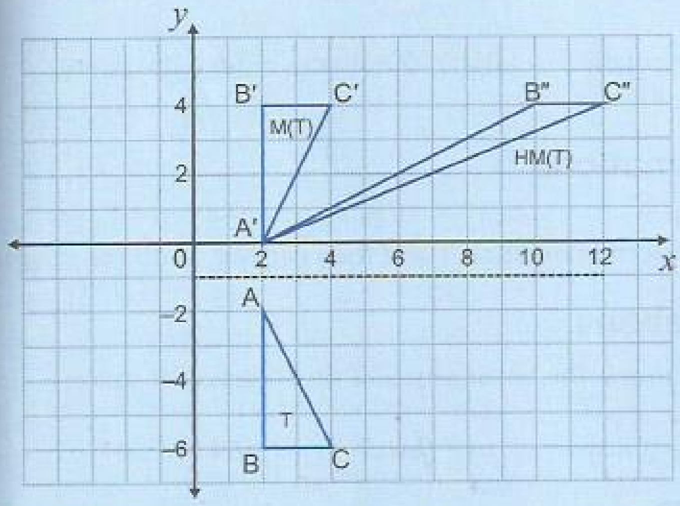

Figure 5.40

The order of combined transformations

Fig. 5.40 shows ΔT and its image M(T) after a reflection in the line $y = -1$. HM(T) is the final image where M(T) has been sheared by H. These transformations can be written as HM(T).

Note

HM(T) means first apply transformation M (reflection) and then apply transformation H (shear) to that.
The order in which you do the transformations is very important. $HM(T) \neq MH(T)$.

Fig. 5.41 shows how the outcome of MH(T) is different from that of HM(T).

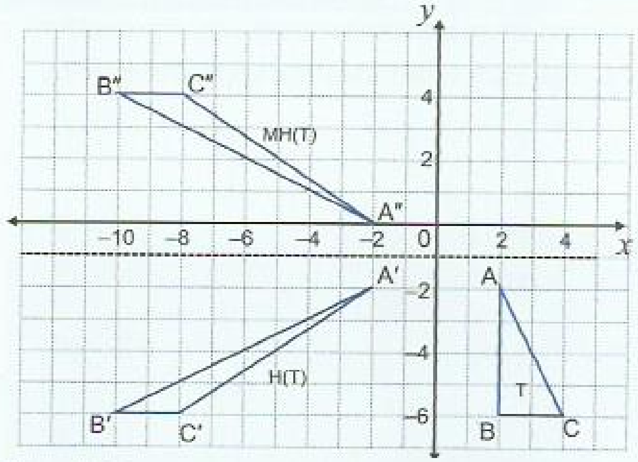

Figure 5.41

The figure shows the transformation of MH(T) coordinates of final image are A"(–2, 0), B"(–10, 4), C"(–8, 4)

Repeated transformations

$M^2(T) = MM(T)$

MM(T) means perform transformation M on T and then perform M on the image.

Inverse transformations

The inverse of a transformation is the transformation which takes the image back to the object.

If R is a 2 × 2 matrix which maps P on to P' then R(P) = P' and inverse $R^{-1}(P') = P$.

Worked example 12

1 ΔPQR has vertices at P(1, 1), Q(1, 3) and R(2, 3). Find the image of ΔPQR if it is first reflected in the line $y = x$ and then translated by vector $T = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$.

2 Quadrilateral ABCD with vertices at A(0, 2), B(–2, 4), C(2, 7) and D(2, 2) is first rotated through 180° about the origin and then sheared by the operator $H = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$. Find the vertices of the final figure.

3 Each of the following equations represents a transformation.

a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 5 \\ -2 \end{pmatrix}$

b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

i) Describe each transformation in words.

ii) Find the image of point Q(2, 1) for each of the transformations.

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d inverse

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or $T = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$.
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d by the operator

rmations.

Worked example 12 (continued)

4 Quadrilateral ABCD has vertices A(0, 0), B(0, 2) C(3, 2) and D(3, 0). Quadrilateral A'B'C'D' has vertices A'(0, 0), B'(0, 2), C'(9, 2) and D'(9, 0). Quadrilateral A'B'C'D' is the image of ABCD under a transformation represented by a matrix of the form $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$.

a) Find the transformation matrix.

b) Find the matrix which will transform quadrilateral A'B'C'D' back to ABCD.

c) Compare the two matrices.

Answers

1 Reflection in $y = x$ matrix $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Translation matrix $T = \frac{5}{-3}$

TM(PQR)

First, reflect by M

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 1 & 2 \end{pmatrix}$$

Second, translate by T

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 8 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \end{pmatrix}$$

The coordinates of the final image of ΔPQR are P"(6, −2), Q"(8, −2) and R"(8, −1)

2 HR represents a combination of the transformations; rotation R (applied first) and shear H (applied second).

$$HR = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -5 & -1 \end{pmatrix}$$

$$HR(ABCD) = \begin{pmatrix} -1 & 0 \\ -5 & -1 \end{pmatrix}\begin{pmatrix} 0 & -2 & 2 & 2 \\ 2 & 4 & 7 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 2 & -2 & 2 \\ -2 & 6 & -17 & -12 \end{pmatrix}$$

The resulting vertices are A(0, −2), B(2, 6), C(−2, −17) and D(−2, −12).

3 a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 5 \\ -2 \end{pmatrix}$

i) First, the enlargement matrix $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ acts on (x, y).
Second, the result is then translated by vector $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$

ii) Substitute $x = 2$ and $y = 1$ in the equation.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

$\begin{pmatrix} -4 \\ -2 \end{pmatrix} + \begin{pmatrix} 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$ ∴Q' is the point (1, −4)

Worked example 12 (continued)

b) $\begin{pmatrix}x'\\y'\end{pmatrix}=\begin{pmatrix}3&0\\0&2\end{pmatrix}\begin{pmatrix}0&-1\\-1&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$

i) First, the reflection matrix $\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$ acts on (x, y).
Second, the result is stretched by matrix $\begin{pmatrix}3&0\\0&2\end{pmatrix}$.

ii) Substitute $x = 2$ and $y = 1$ in the equation.

$$\begin{pmatrix}x'\\y'\end{pmatrix}=\begin{pmatrix}3&0\\0&2\end{pmatrix}\begin{pmatrix}0&-1\\-1&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

$\begin{pmatrix}0&-3\\-2&0\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}-3\\-4\end{pmatrix}$ ∴ Q' is the point $(-3, -4)$.

4 a) $\begin{pmatrix}a&0\\0&b\end{pmatrix}\begin{pmatrix}0\\2\end{pmatrix}=\begin{pmatrix}0\\2\end{pmatrix}$ ①

$\begin{pmatrix}a&0\\0&b\end{pmatrix}\begin{pmatrix}3\\2\end{pmatrix}=\begin{pmatrix}9\\2\end{pmatrix}$ ②

From ①: $2b = 2$
$b = 1$

From ②: $3a = 9$
$a = 3$

The transformation matrix is $\begin{pmatrix}3&0\\0&1\end{pmatrix}$, a stretch.

b) Let $\begin{pmatrix}3&0\\0&1\end{pmatrix}$ be matrix S.
If S maps ABCD onto A'B'C'D', then inverse S^{-1} will map A'B'C'D' back to ABCD, the original quadrilateral.

$S^{-1}=\frac{1}{3}\begin{pmatrix}1&0\\0&3\end{pmatrix}$

$\therefore S^{-1}=\begin{pmatrix}\frac{1}{3}&0\\0&1\end{pmatrix}$ will map A'B'C'D' back to ABCD

c) $S=\begin{pmatrix}3&0\\0&1\end{pmatrix}$, let T represent S^{-1}

$T=\begin{pmatrix}\frac{1}{3}&0\\0&1\end{pmatrix}$, $T^{-1}=3\begin{pmatrix}1&0\\0&\frac{1}{3}\end{pmatrix}=\begin{pmatrix}3&0\\0&1\end{pmatrix}$

∴ each matrix is the inverse of the other.

Activity 8

1 M is a reflection in the y-axis and H is a shear represented by $\begin{pmatrix}1&-2\\0&1\end{pmatrix}$.
Calculate the coordinates of the point that P(−2, 5) is mapped onto under the following transformations:

a) HM(P) b) MH(P) c) H^2(P) d) H^{-1}(P)

2 Write down the matrix which represents each of the following:

a) Rotation of 180° about (0, 0)
b) Enlargement, centre (0, 0), scale factor −2
c) Reflection in the y-axis
d) Rotation −90° about (0, 0)
e) Reflection in $y = -x$

Activity 8 (

f) Shear
g) Stretch
h) Rotatic
3 R denotes
M denotes
E denotes
T denotes
Draw ΔABC
Find the in
transforma
a) TR(ABC
b) RM(AB
c) ET(ABC
d) RE(ABC
e) MT^2(AB
f) TRM(AB
4 Each of the
(x', y') is the
$\begin{pmatrix}x'\\y'\end{pmatrix}=\begin{pmatrix}-1\\0\end{pmatrix}$
$\begin{pmatrix}x'\\y'\end{pmatrix}=\begin{pmatrix}2&0\\0&5\end{pmatrix}$
$\begin{pmatrix}x'\\y'\end{pmatrix}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$
a) Describe
b) Find the
c) The coor
B(a, b) u
5 ΔPQR has ve
onto ΔP'Q'R'
transformati
a) the trans
b) the matri

p A'B'C'D' back to

by $\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$.
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d) H^{-1}(P)
ing:

Activity 8 (continued)

f) Shear of factor 3, in the y-direction with y-axis invariant
g) Stretch of factor –5 in the x-direction with y-axis invariant
h) Rotation +90° about (0, 0)

3 R denotes a rotation of 180°, centre (0, 1).
M denotes a reflection in the line $y = 0$.
E denotes an enlargement, scale factor –2, centre (0, 0).
T denotes a translation matrix $\begin{pmatrix} -5 \\ 1 \end{pmatrix}$.
Draw ΔABC with vertices A(2, 2), B(6, 2) and C(6, 4).
Find the images of ΔABC under the following combinations of transformations.

a) TR(ABC)
b) RM(ABC)
c) ET(ABC)
d) RE(ABC)
e) MT^2(ABC)
f) TRM(ABC)

4 Each of the following equations represents a transformation in which (x', y') is the image of the point (x, y).

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ -7 \end{pmatrix}$$

a) Describe the transformation in words.
b) Find the image of the point A(–2, 5) for each transformation.
c) The coordinates of point B' are (3, –4). Find the coordinates of the point B(a, b) under each transformation.

5 ΔPQR has vertices at points P(0, 4), Q(2, 1) and R(3, 5). ΔPQR is mapped onto ΔP'Q'R'. Vertices at P'(0, 12), Q'(–4, 3) and R'(–6, 15) are given by a transformation matrix of the form $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Find

a) the transformation matrix
b) the matrix which will map ΔP'Q'R' back onto ΔPQR.

SUB-TOPIC 9

Find area scale factor of a stretch by determinant method

As we look at a stretch, remember that:

- a stretch has an invariant line and points on the invariant line do not move
- the distance each point moves is proportional to its distance from the invariant line

Worked example 13

Find the matrix for a stretch S parallel to the x-axis with the y-axis invariant, under which the point Q(2, 1) is mapped onto Q'(8, 1).

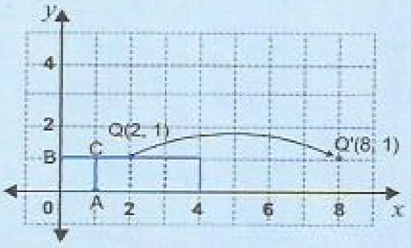

Figure 5.42

Answer

Fig. 5.42 shows Q(2, 1) and its image Q'(8, 1).

To find the matrix of the stretch S, take A as (1, 0) and B as (0, 1).

Q(2, 1) is 2 units from the invariant line and has moved 6 units to Q'(8, 1)

Then A(1, 0) which is 1 unit from the invariant line will move 3 units to A'(4, 0).

OB is invariant, therefore the coordinates for B remain (0, 1).

Hence the transformation matrix S is given by:

$$S = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$$

From Fig. 5.42, note that the unit square OACB has been mapped onto rectangle OA'C'B. Now the rectangle OA'C'B has an area of 4 squares, and so the area of square OABC has been multiplied by 4.

The determinant of the matrix $\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} = 4$

Therefore in a stretch transformation, the determinant gives the multiplier for the areas.

The determinant of a matrix is calculated from the coordinates of the matrix.

We use |S| to represent the determinant of the matrix S.

For a 2 × 2 matrix, the determinant is calculated as follows:

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

|S| or det(S) = ad – bc

Worked exa

1 ΔPQR has v

y-axis is inv

stretch, find

a) the stret

b) the imag

c) the ratio

2

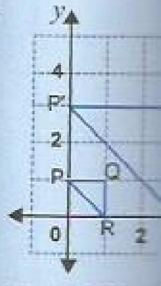

Figure 5.43

ΔPQR is map

factor of 3.

a) Find the

b) Calculate

3 In Fig. 5.44,

the x-axis inv

a) Find the

the shear.

b) Find the

of OABC

of OAB'C'

Answers

1 a) Since A(1,

Stretch fac

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Determina

b) The image

Since A(1,

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tes of the matrix.

Worked example 14

1 ΔPQR has vertices P(2, 1), Q(4, 1) and R(5, 3). S is a stretch such that the *y*-axis is invariant. Given that A(1, 0) is mapped onto A'(3, 0) under this stretch, find:

a) the stretch matrix S

b) the image of ΔPQR under this transformation

c) the ratio of the area of ΔPQR to the area of ΔP'Q'R'.

2

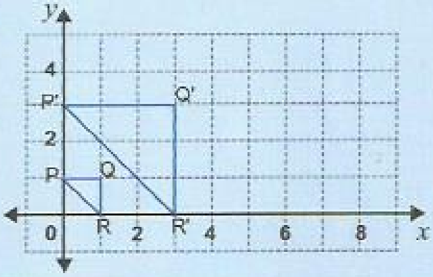

Figure 5.43

ΔPQR is mapped onto ΔP'Q'R' by an enlargement with centre O and a scale factor of 3.

a) Find the enlargement matrix.

b) Calculate the ratio of the area of ΔPQR to the area of ΔP'Q'R'.

3 In Fig. 5.44, the square unit OABC is mapped onto OAB'C' by a shear with the *x*-axis invariant.

a) Find the matrix of the shear.

b) Find the ratio of area of OABC to the area of OAB'C'.

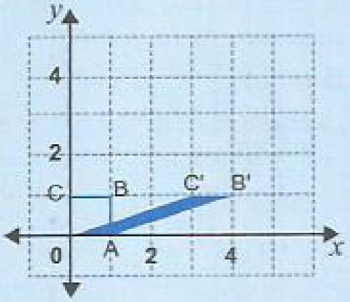

Figure 5.44

Answers

1 **a)** Since A(1, 0) is mapped onto A'(3, 0) and the *y*-axis is invariant.

$$\text{Stretch factor} = \frac{\text{distance of image from invariant line}}{\text{distance of object from invariant line}}$$

Then matrix $S = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$

Determinant of $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = 3$

b) The image of ΔPQR.

Since A(1, 0) has moved 2 units to A'(3, 0), then P(2, 1) will move 4 units, as it is 2 units from the invariant line.

Worked example 14 (continued)

And P' is (6, 1)
Similarly Q' is (12, 1) and R' is (15, 3)
or by multiplying the matrices:

$$\begin{array}{cc} & \begin{array}{ccc} P & Q & R \end{array} \quad \begin{array}{ccc} P' & Q' & R' \end{array} \\ & \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 & 5 \\ 1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 12 & 15 \\ 1 & 1 & 3 \end{pmatrix} \end{array}$$

That is, P'(6, 1), B'(12, 1) and C'(15, 3)

c)

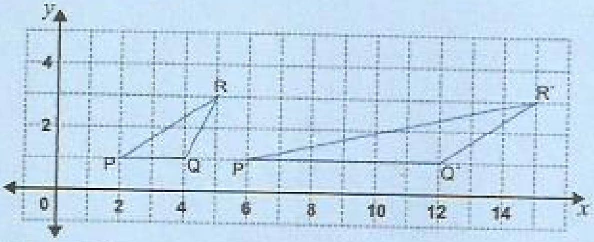

Figure 5.45

The ratio of the base PQ to P'Q' is 2 : 6 = 1 : 3.
Area of image P'Q'R' = det (matrix) × area of PQR

$\det \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = 3$

∴ area of P'Q'R' = 3 × area of PQR.
∴ the ratio of area of ΔPQR: area of ΔP'Q'R' = 1 : 3

2 a) The centre of enlargement is the origin (0, 0) and the scale factor is 3.
The matrix of enlargement is $E = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$.

b) The determinant of $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ is $3^2 = 9$.
∴ the ratio of the area of ΔPQR : ΔP'Q'R' is 1 : 9.

Note

Shear does not change the area of a shape, and it will always have a determinant of 1.

3 a) Scale factor = $\frac{BB'}{AB} - \frac{2}{1} = 2$
So the shear matrix is $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

b) The det $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = 1$
Since the determinant is 1 it means the area has not changed.
∴ the area of OABC = area of OAB'C' and so the ratio is 1:1.

Activity 9

1 The stretch S is such that the y-axis is invariant. ΔPQR is mapped onto ΔP'Q'R' under this transformation, and the image of P(3, 5) is P'(6, 5).
- a) Find the scale factor of the stretch.
- b) Find the coordinates of the images of P, Q and R.
- c) Draw ΔPQR and its image ΔP'Q'R' on Cartesian plane.
- d) Find the matrix S.
- e) Find the ratio of the area of ΔPQR to the area of ΔP'Q'R'.

Activity 9 (c

2 An enlarger
a) Find the
b) Find the
c) Calculat
the area

3 Fig. 5.47 sho
which maps
a) the centr
b) the scale
c) the ratio
the area
d) the area
ΔPQR is 2

4 A rhombus P
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a) Find the c
b) Find the a
c) Calculate

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ar does not change
area of a shape,
it will always have
eterminant of 1.

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mapped onto
) is P'(6, 5).

Activity 9 (continued)

2 An enlargement E maps ΔT onto ΔR.
  a) Find the centre of enlargement.
  b) Find the scale factor of the enlargement.
  c) Calculate the ratio of the area of ΔT to the area of ΔR.

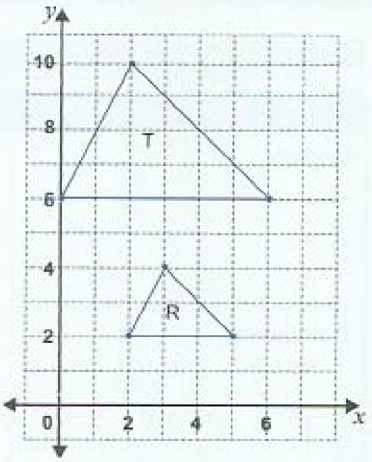


Figure 5.46

3 Fig. 5.47 shows an enlargement E which maps ΔPQR onto ΔPST. Find:
  a) the centre of enlargement
  b) the scale factor of the enlargement
  c) the ratio of the area of ΔPQR to the area of ΔPST
  d) the area of ΔPST, if the area of ΔPQR is 24 cm².

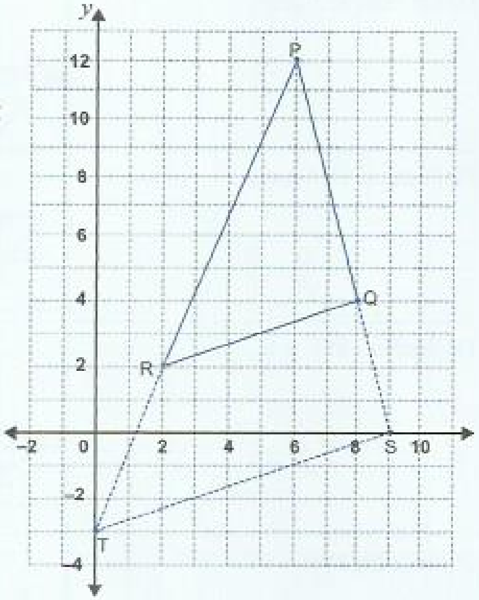
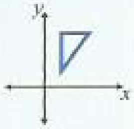
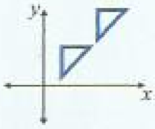
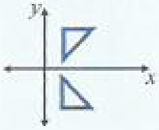
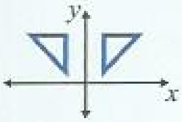
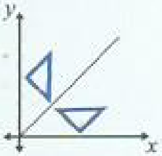
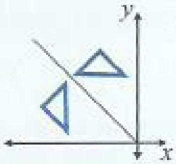
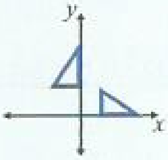


Figure 5.47

4 A rhombus PQRS with vertices at (1, 4), (2, 1), (3, 4) and (2, 7) respectively is mapped onto P'Q'R'S' by a transformation matrix $\begin{pmatrix}2 & 0\\0 & 1\end{pmatrix}$.
  a) Find the coordinates of P'Q'R'S', the image of PQRS.
  b) Find the area of P'Q'R'S'.
  c) Calculate the area of PQRS if the area of P'Q'R'S' is 18 cm².

TOPIC 5 Summary, revision and assessment

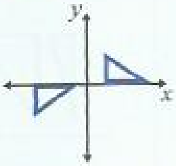
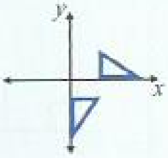
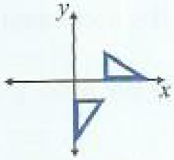
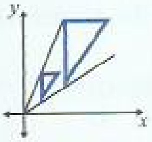
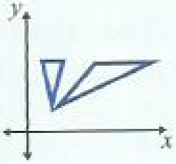
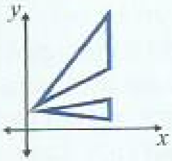
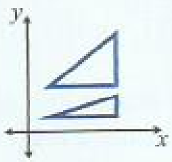
Summary

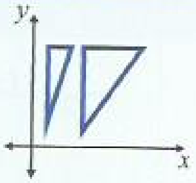
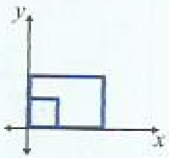
Transformation	Sketch	Matrix
Identity		$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
Translation		$\begin{pmatrix} a \\ b \end{pmatrix}$
Reflection in x-axis		$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
Reflection in y-axis		$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
Reflection in $y = x$		$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
Reflection in $y = -x$		$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$
Rotation anticlockwise 90° centre (0, 0)		$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Transfo
Rotation 180° centre (0, 0)
Rotation 270° centre (0, 0)
Rotation clockwi centre (0, 0)
Enlargement centre (0, 0)
Shear x-axis inva
Shear y-axis inva
Stretch x-axis in

Matrix
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$\begin{pmatrix} a \\ b \end{pmatrix}$
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Summary, revision and assessment (continued)

Transformation	Sketch	Matrix
Rotation 180° centre (0, 0)		$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
Rotation 270° centre (0, 0)		$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
Rotation clockwise 90° centre (0, 0)		$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
Enlargement centre (0, 0)		$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$
Shear x-axis invariant		$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$
Shear y-axis invariant		$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$
Stretch x-axis invariant		$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$

Transformation	Sketch	Matrix
Stretch y-axis invariant		$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$
Stretch 2-way stretch in x and y-directions		$\begin{pmatrix} h & 0 \\ 0 & k \end{pmatrix}$

- If a transformation is a shear, the area remains unchanged.
- For other transformations:
  - If A is the original area, then the area of the image = det(matrix) × A

Revision

1 Using base vectors, describe the transformation represented by the following matrices.

a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ b) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ c) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ d) $\begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}$

e) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ f) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ g) $\begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$

2 P is a rotation of 90° anticlockwise about (0, 0). Q is a reflection in the line $x = 3$. R is a translation which maps (2, −1) onto (−3, −1) Find the image of the point (1, 2) under:

a) P b) P^2 c) QR d) R^{-1} e) PQR f) $R^{-1}Q^{-1}P^{-1}$

3 ΔPQR with vertices P(2, 2), Q(4, 2) and R(3, 4) is mapped onto a triangle with vertices P'(−2, −1), Q'(0, −1) and R'(−1, 1).

a) Draw and label ΔPQR and its image ΔP'Q'R'.

b) Describe fully the transformation which maps ΔPQR onto ΔP'Q'R'.

4 ΔEFG with vertices E(1, 1), F(2, 1) and G(2, 2) is mapped onto ΔE'F'G' with vertices E'(1, −3), F'(2, −3) and G'(2, −6) by a stretch S. Find:

a) the matrix S b) the scale of the stretch factor

5 ΔSTU with vertices S(0, 1), T(1, 1), U(1, 3) is mapped onto S'(4, 1), T'(3, 1) and U'(3, 3).

a) Find the equation of the mirror line of this reflection.

b) Find the image of ΔSTU under a reflection in the y-axis.

c) Find the translation that maps this image onto ΔS'T'U'.

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6 A'(0, 0), B'(
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b) Find th

Assessme

1 ΔP has verti
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b) The tran
c) ΔQ is m
enlargem
the cent
d) ΔQ is m
rotation
of rotati
e) The mat
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of ΔT.
ii) Descr
by $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

2 Fig. 5.49 sho
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c) ΔE is map
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i) the eq
invariant
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d) ΔE is map
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2 with (0,
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Matrix
$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$
$\begin{pmatrix} h & 0 \\ 0 & k \end{pmatrix}$

matrix) × A

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d) $\begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}$

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ind the image of the

f) $R^{-1}Q^{-1}P^{-1}$

nto a triangle with

nto ΔP'Q'R'.
nto ΔE'F'G' with

etch factor
S'(4, 1), T'(3, 1) and

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6 A'(0, 0), B'(8, 9) and C'(16, 3) are the images of A(0, 0), B(2, 3) and C(4, 1) under a transformation represented by a matrix of the form $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$.

  a) Find the transformation matrix.

  b) Find the matrix that would transform ΔA'B'C' back to ΔABC.

Assessment

1 ΔP has vertices (–2, 3), (–2, 6) and (–4, 3).

  a) Using a scale of 1 cm to represent 1 unit on each axis, draw axes for values of x and y in the range $-8 \le x \le 8$ and $-8 \le y \le 8$. Draw and label ΔP.

  b) The translation $T = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$ maps ΔP onto ΔQ. Draw and label ΔQ.

  c) ΔQ is mapped onto ΔR by an enlargement $E = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ with O(0, 0) as the centre. Draw and label ΔR.

  d) ΔQ is mapped onto ΔS by a clockwise rotation of 90° with (1, 4) as the centre of rotation. Draw and label ΔS.

  e) The matrix $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ represents the transformation which maps ΔQ onto ΔT.

    i) Find the coordinates of the vertices of ΔT.

    ii) Describe the transformation represented by $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ in full.

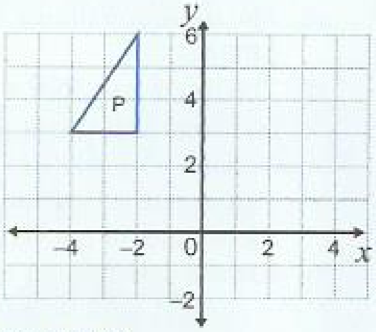

Figure 5.48

2 Fig. 5.49 shows triangles E, F, G and H.

  a) A translation T maps ΔE onto ΔF. Find the column vector for T.

  b) Describe a transformation which maps ΔE onto ΔG in full.

  c) ΔE is mapped onto ΔH by a shear H. Write down:

    i) the equation of the invariant line

    ii) the shear factor.

  d) ΔE is mapped onto ΔI by an enlargement of factor 2 with (0, 1) as the centre. Draw and label ΔI.

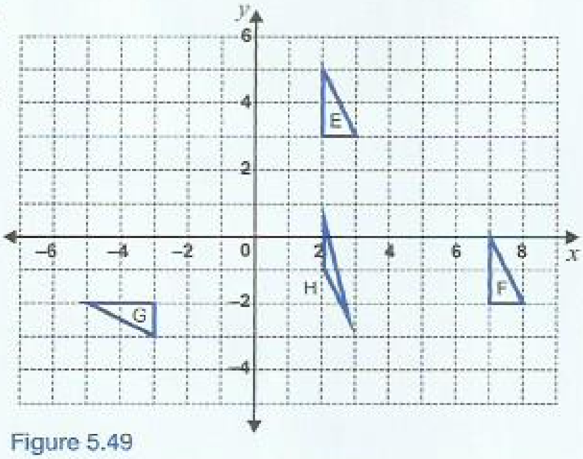

Figure 5.49

Summary, revision and assessment (continued)

3 The point (x', y') is the image of the point (x, y) after a combination of transformations given by:

$$\begin{pmatrix}x'\\y'\end{pmatrix}=\begin{pmatrix}1&4\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}2\\5\end{pmatrix}$$

  a) Find the coordinates of O', the image of the point O(0, 0).
  b) Find the coordinates of A', the image of the point A(2, 2).
  c) If B'(6, 8) is the image of B(a, b), form two equations and solve for a and b.

4 ΔA with vertices (2, 4), (4, 4) and (4, 1) is mapped onto ΔB with vertices (6, 12), (12, 12) and (12, 3).
  a) **i)** Draw and label triangles A and B.
    ii) Describe the transformation that maps ΔA onto ΔB in full.
  b) R is a clockwise rotation of 90° about the origin. Draw and label ΔR(A).
  c) The transformation T is the translation $\begin{pmatrix}-7\\3\end{pmatrix}$. Draw and label ΔT(A) and ΔRT(A).
  d) The single transformation M is represented by the matrix $\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$. Draw and label ΔM(A) and describe the transformation M in full.

5 A unit square has vertices A(0, 0), B(1, 0), C(1, 1), and D(0, 1).
  a) This square is transformed under the matrix $\begin{pmatrix}7&2\\1&4\end{pmatrix}$. Write down the coordinates of the vertices of the image.
  b) Use the determinant to find the area of parallelogram ABCD.

TOPIC
6

Sub-
Introduction to Eart
Small and great circ
Latitudes and longit
Speed in knots and

Starter activit

Here are two diagr
in Africa while Fig
radius of 6 370 km

Figure 6.1a

1 Name the lines
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5 What is the le
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6 Estimate how f

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TOPIC 6

Earth geometry

Sub-topic	Specific Outcomes
Introduction to Earth geometry	- Explain the concept of Earth geometry.
Small and great circles	- Distinguish between small and great circles.
Latitudes and longitudes	- Calculate distance along parallels of latitudes and longitudes in kilometres and nautical miles. - Calculate the shortest distance between two places on the surface of the Earth.
Speed in knots and time	- Calculate speed in knots and time.

Starter activity

Here are two diagrams of the Earth: Fig. 6.1a highlights the position of Zambia in Africa while Fig. 6.1b is a blank representation of the Earth. The Earth has a radius of 6 370 km. Use the diagrams to answer the questions.

Figure 6.1a

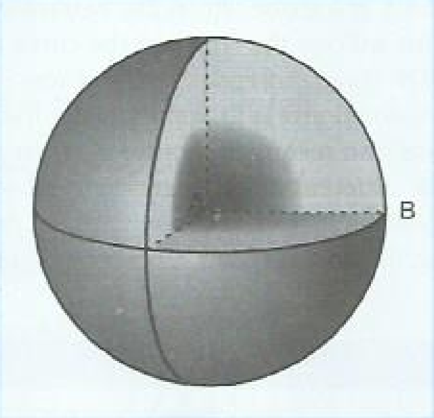

Figure 6.1b

1 Name the lines X, Y and Z.
2 Name the points V and W.
3 What do you think the point A is?
4 What does the line AB represent and what is its actual length?
5 What is the length of the straight line WV (which goes through the centre of the Earth)?
6 Estimate how far the centre of Zambia is from the Equator.

SUB-TOPIC 1 Introduction to Earth geometry

Earth geometry and Euclidean geometry

The geometry that you have studied until now, such as circle theorems and congruency, is based on Euclidean geometry. This kind of geometry uses the way lines and angles interact on a flat plane.

When we work with relatively small distances on the Earth's surface, such as the distance between towns in Zambia, we use Euclidean geometry for our calculations. When we measure distances that an aeroplane flies between two countries that are far apart on the Earth's surface, we need to go around the curve of the Earth from one place to another. Here we need to use Earth geometry for our calculations.

Figure 6.2a

Think about a direct or straight-line distance between a place in Zambia and a city in China. On a map (Fig. 6.2a) the route from Zambia to China looks like a straight line. However, when you look at a globe (Fig. 6.2b) you realise that the line follows the shape of the curve of the Earth in three dimensions. The three-dimensional line is longer than the flat line.

Figure 6.2b

This also means that other geometry rules will be different in Earth geometry. If we draw a triangle on the Earth's surface, because the surface is curved, we can draw a triangle with two 90° angles! In Fig. 6.3 angles B and C are each equal to 90°.

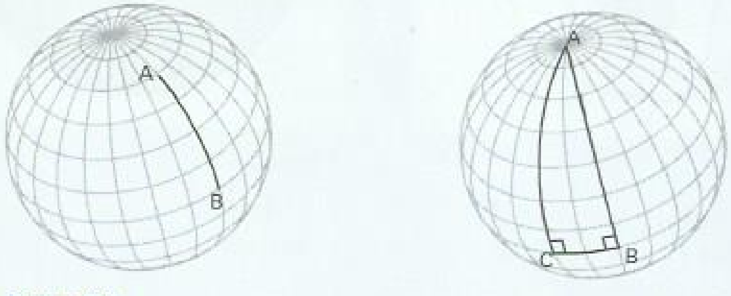

Figure 6.3

The uses of Earth geometry

We use Earth geometry for locating places on the Earth's surface and determining distances and universal time between such places.

In reality the
and South Poles)
valleys, and so o
In Earth geometr
- The Earth is sp
- The surface of

The propert

- A sphere is a p
dimensional o
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of the sphere.
are the same d
- The distance r
- The diameter i
passing throug
it is equal to tw
- A sphere is an
is why objects
- Distances betw
length of arcs
circle circumfe

Remember these

Useful formulae for

Area of a circle = πr
Length of circumfer
Area of a circle sect
Length of an arc of circle sector with angle $\theta = \frac{\theta}{180°}\pi r$

In reality the Earth is not quite spherical (it is slightly flattened at the North and South Poles). The surface of the earth is not even, as there are mountains and valleys, and so on.

In Earth geometry we make these assumptions:

- The Earth is spherical and has a radius of 6 371 km.
- The surface of the earth is even.

The properties of a sphere

- A sphere is a perfectly round three-dimensional object.
- The point O in the middle is the centre of the sphere. All points on the surface are the same distance from the centre.
- The distance r is the radius of the sphere.
- The diameter is the longest straight line passing through the centre of the sphere; it is equal to twice the radius.
- A sphere is an object that has the smallest surface area for a given volume. This is why objects in nature are often spherical.
- Distances between points on a sphere can be calculated by calculating the length of arcs formed between the points. We treat these distances as part of circle circumferences.

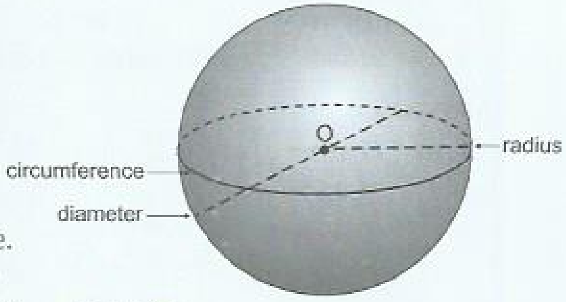


Figure 6.4

Remember these parts of the circle:

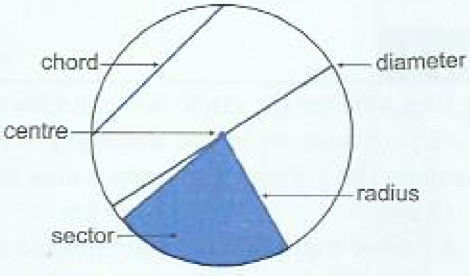
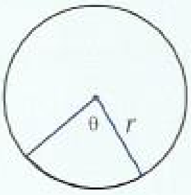


Figure 6.5

Useful formulae for circles and spheres

Circle	Sphere
Area of a circle = πr^2	Surface area of a sphere = $4\pi r^2$
Length of circumference of a circle = $2\pi r$	Volume of a sphere = $\frac{4}{3}\pi r^3$
Area of a circle sector with angle $\theta = \frac{\theta}{360^\circ}\pi r^2$	
Length of an arc of a circle sector with angle $\theta = \frac{\theta}{180^\circ}\pi r$ 	

Worked example 1

Using the formulae given for circles and spheres, calculate the following. Round off your answers to two decimal places.

1 The length of the arc of a circle sector with radius 5 cm with these angles.
   a) 90° b) 40° c) 110°

2 The area of the circle sector with radius 120 m and the these angles.
   a) 40° b) 60° c) 125°

3 The surface area of a quarter of a sphere with radius 16 km.

Answers

1 a) Length $= \frac{\theta}{180°}\pi r = \frac{90°}{180°}\pi(5) = 7.85$ cm

b) Length $= \frac{\theta}{180°}\pi r = \frac{40°}{180°}\pi(5) = 3.49$ cm

c) Length $= \frac{\theta}{180°}\pi r = \frac{110°}{180°}\pi(5) = 9.60$ cm

2 a) Area $= \frac{\theta}{360°}\pi r^2 = \frac{40°}{360°}\pi(120)^2 = 5\ 026.55\ \text{m}^2$

b) Area $= \frac{\theta}{360°}\pi r^2 = \frac{60°}{360°}\pi(120)^2 = 7\ 539.82\ \text{m}^2$

c) Area $= \frac{\theta}{360°}\pi r^2 = \frac{125}{360}\pi(120)^2 = 15\ 707.96\ \text{m}^2$

3 Surface area $= \frac{1}{4}(4\pi r^2) = \pi(16)^2 = 804.25\ \text{km}^2$

Activity 1

Use the formulae for the circle and the sphere to answer the following questions. Round off your answers to two decimal places.

1 Calculate the volume and surface area of spheres with the following radii.
   a) 15 cm b) 3 km c) 42 m

2 a) A rubber football has an outside diameter of 22 cm. The thickness of the rubber is 0.5 cm. What is the volume of the rubber itself to the nearest cubic centimetre (cm^3)?
   b) The same rubber football is sliced in half. What is the surface area of the rubber of one of the hemispheres?

3 The radius of the Earth is approximately 6 370 km.
   a) Calculate the circumference of the Earth at its widest position.
   b) What is the distance from the North Pole to the South Pole directly through the centre of the Earth?
   c) What is the distance from the North Pole to the South Pole along the surface of the Earth?
   d) What is the surface area of the Earth?
   e) What is the volume of the Earth?

Activity 1 (c

f) The Eart
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4 Find the len
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   a) 80°

5 Find the are
   a) 75°

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2 m

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Pole along the

Activity 1 (continued)

f) The Earth's atmosphere is about 480 km thick, but most of the gases in the atmosphere are concentrated within 16 km of the surface of the Earth. Calculate the volume of this concentrated part of the atmosphere.

4 Find the length of the arc of the circle sector with radius 2 000 km and these angles.

a) 80° b) 40° c) 110°

5 Find the area of the circle sector with radius 5 000 km and these angles.

a) 75° b) 32° c) 200°

SUB-TOPIC 2 Great and small circles

The orientation of the Earth

We use a particular model of the Earth for our study of Earth geometry. This model is based on two fixed points: the North and South Poles of the Earth's axis, around which it rotates. This model comes from a formal decision to regard the North Pole as the one around which the planet rotates anticlockwise.

Fig. 6.6 shows a model of the Earth.

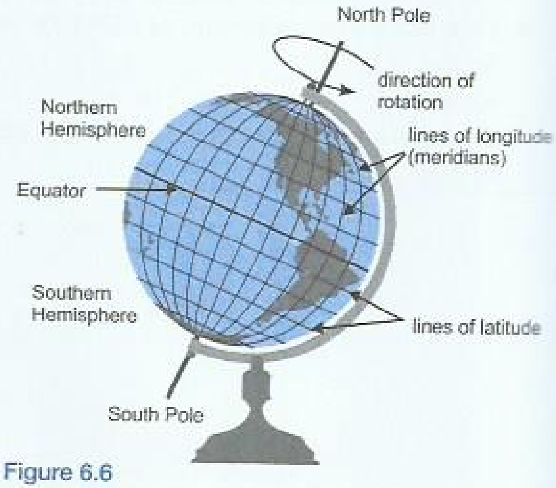

Figure 6.6

On a globe of the Earth, we commonly refer to its two halves as the Southern Hemisphere and the Northern Hemisphere. However, the sphere could be sliced at an infinite number of different places to produce identical hemispheres.

New word

hemisphere: half of a sphere (as if the sphere has been sliced in half right through the middle)

An introduction to latitude and longitude

Before we learn about great and small circles of the Earth it would help us if we knew something about concepts such as latitude and longitude.

To measure accurately the position of any place on the surface of the Earth, a grid system has been set up. It pinpoints a location by using two coordinates: latitude and longitude.

- Lines of latitude are imaginary parallel lines that run from east to west around the Earth's surface. The longest of these is called the Equator and is the only line of latitude with a radius of 6 370 km.
- Lines of longitude represent east-west location. They are shown by a series of north-south running lines that all meet at the North and South Poles. They are the widest apart at the Equator. Lines of longitude are also called meridians.

New word

meridian: an imaginary line forming a circle that passes through the Earth's North and South Poles

Great circles

- A great circle of the Earth is a circle on the surface of the Earth whose radius is equal to that of the Earth. This means that the Equator is a great circle.

- All lines of lor
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- All great circle
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Small circles

- A small circle
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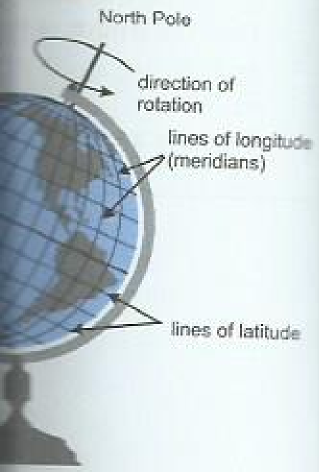

New word

hemisphere: half of a sphere (as if the sphere has been sliced in half right through the middle)

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New word

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rming a circle that passes
rough the Earth's North
d South Poles

e Earth whose radius is
is a great circle.

- All lines of longitude are great circles passing through the North and South Poles.
- There are an infinite number of great circles of the Earth as they can be drawn at any point on the Earth's surface.
- All great circles of the Earth are exactly the same size: they divide the Earth into two equal hemispheres, no matter where they are.

Small circles

- A small circle of the Earth is a circle on the surface of the Earth whose radius is less than the radius of the Earth.
- All lines of latitude, except the Equator, are small circles.
- There are also small circles which are not latitudes.

The shortest surface distance between any two points on a sphere is the length of the arc along the great circle through those points.

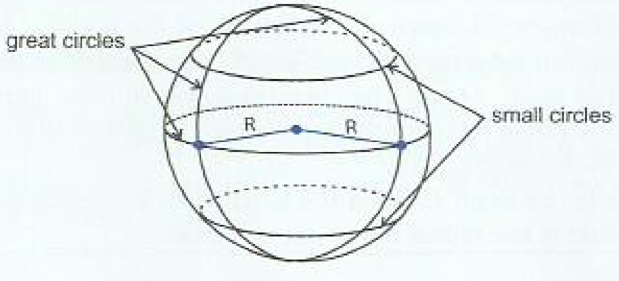

Figure 6.7

The size of a small circle depends on its position. Fig. 6.8 shows what the globe would look like if it was sliced at the Equator and at two of its small circles parallel to the Equator. The cross-section of each slice is a circle with a different radius. The size of the radius depends on the distance of the small circle from the Equator. The further away from the Equator the small circle is, the smaller the radius of the cross-section.

Figure 6.8

Activity 2

1 a) In Fig. 6.9, write down the letters that indicate great circles.
  b) Write down the letters of the small circles.

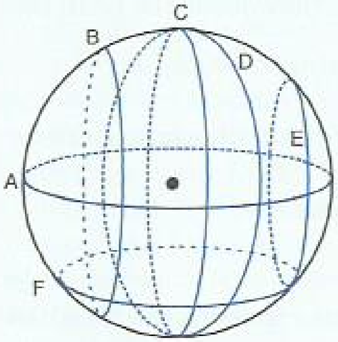

Figure 6.9

2 Look at an atlas. Between which two major latitudes does Zambia lie?

3 The Tropic of Cancer is located at 23.5° north of the Equator and the Tropic of Capricorn lies at 23.5° south of the Equator and runs through the northern part of South Africa. The circumferences of these latitudes are each 36 788 km. Re-arrange the formula for the circumference of a circle to find the radius of these small circles.

4 The Arctic Circle is a small circle of the Earth with a circumference of 17 662 km. What is the radius of the Arctic Circle?

Figure

Locating po

Fig. 6.11 shows pa
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is located by defin
latitude first and t

Figure 6.1

Points A to H repre

- A is 30° east and
- B is 0°, 60° W
- C is 30° S, 30° W

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SUB-TOPIC 3 Latitudes and longitudes

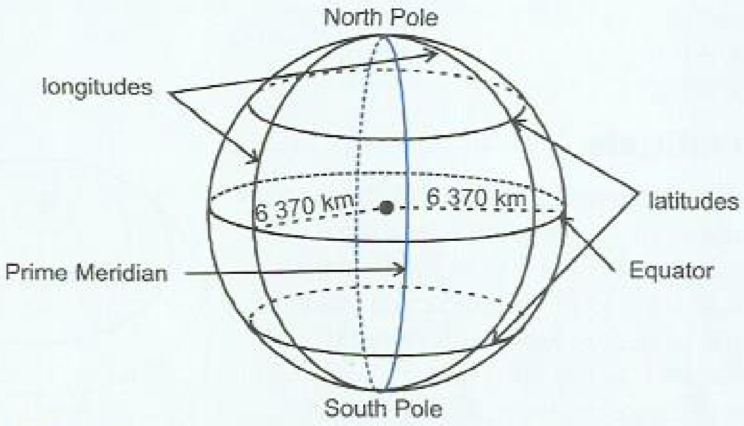

Figure 6.10

Locating positions on the Earth's grid

Fig. 6.11 shows part of the Earth's grid. It shows how far east or west of the Prime Meridian, or how far north or south of the Equator, a place is. Any place on Earth is located by defining its latitude and longitude. The convention is to state the latitude first and then the longitude.

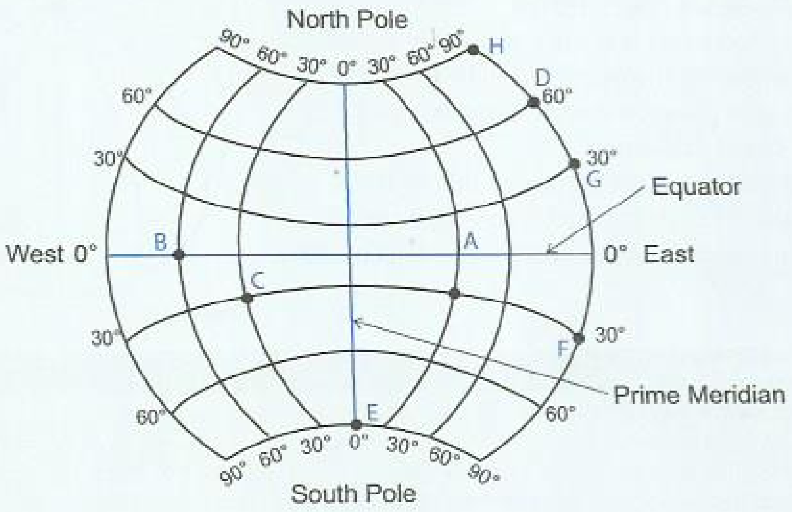

Figure 6.11

Points A to H represent places on Earth. We describe the coordinates as follows:

- A is 30° east and 0° north or south (it is on the Equator): we write this 0°, 30° E.
- B is 0°, 60° W
- C is 30° S, 30° W

- D is 60° N, 90° E
- E is 90° south and 0° west or east (it is on the Prime Meridian): we write this 90° S, 0°.
- F is 30° S, 90° E
- G is 30° N, 90° E
- H is 90° N, 90° E

More on latitude

- Latitude is distance north or south of the Equator.
- The latitude of any given place is its distance, measured in degrees of arc, from the Equator.
- Latitude is numbered in both directions from the Equator, so the Equator is numbered 0° and the Poles 90° N and 90° S.
- Except for the Equator, we write N or S after the number given for the latitude.

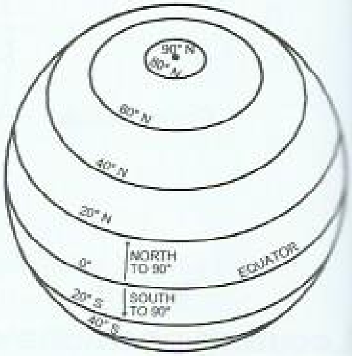

Figure 6.12

More on longitude

- Longitude is distance east or west of the Prime Meridian.
- The longitude of any given place is its distance, measured in degrees of arc, from the Prime Meridian.

Some special meridians

The Prime Meridian is more commonly known as the Greenwich (pronounced 'GREH-nich') Meridian as it passes through Greenwich in London, Great Britain.

The meridian on the opposite side of the Earth to the Greenwich Meridian is known as the 180th Meridian or the International Date Line.

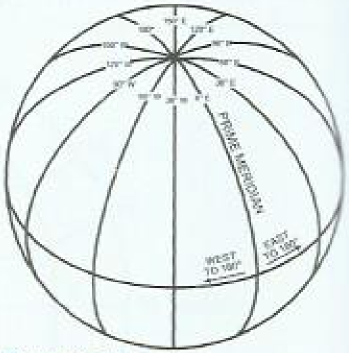

Figure 6.13

Did you know?

The Global Positioning System (GPS) is a system of 27 satellites that orbit (travel around) the Earth. A GPS receiver, like the ones that give directions in a car, on a sports watch, or on a mobile phone, links to four or more satellites and detects the distance to the user. Then it gives the geographical coordinates, using the distance information and using Earth geometry to do the calculations.

Worked exa

Give the coordin
Fig. 6.14. Appro
1 Lukulu
4 Serenje

Figure 6.14

Answers

1 Lukulu 14° S
3 Kasempa 13°
5 Nakonde 9°

Subdividing la

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Worked example 2

Give the coordinates for the following towns in Zambia, using the map in Fig. 6.14. Approximate your answers to the nearest degree.

1 Lukulu
2 Lusaka
3 Kasempa
4 Serenje
5 Nakonde
6 Mambwe

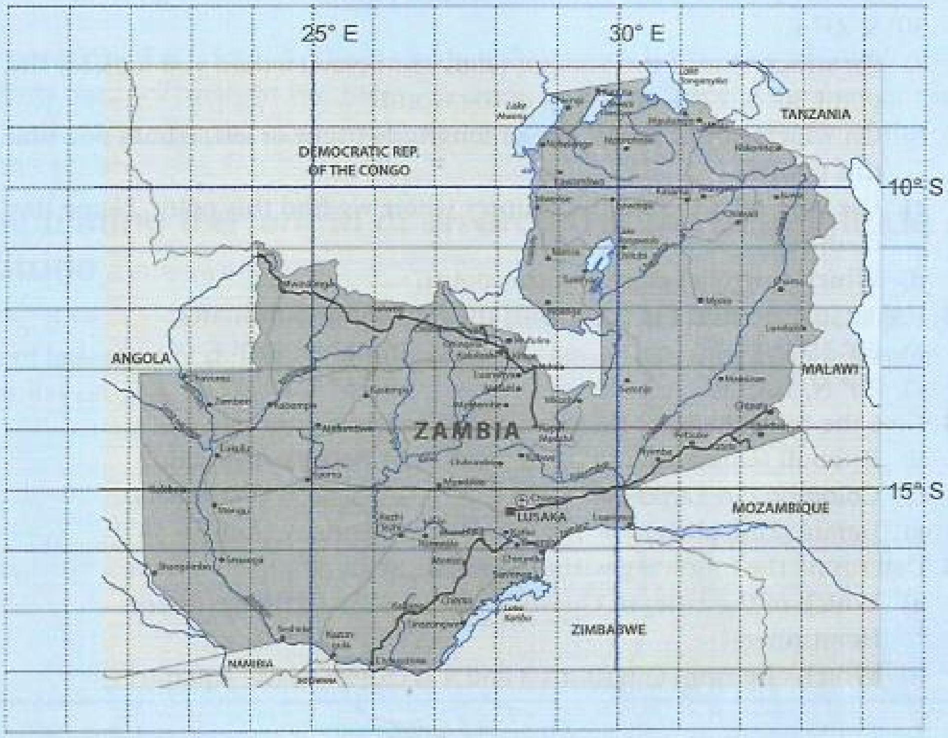

Figure 6.14

Answers

1 Lukulu 14° S, 23° E
2 Lusaka 15° S, 28° E
3 Kasempa 13° S, 26° E
4 Serenje 13° S, 30° E
5 Nakonde 9° S, 33 °E
6 Mambwe 13° S, 32° E

Subdividing latitude and longitude

For more precise measurements, you can include minutes of arc. Minutes of arc are similar to minutes of time in that one minute of latitude is $\frac{1}{60}$th of a degree; there are 60 minutes of arc in 1 degree.

Just as with time, a minute of arc is represented by an apostrophe: '.
For example, a more precise position for Kasempa would be 13° 30' S, 25° 44' E.

Activity 3

1 Use the map in Fig. 6.14. Name the places in Zambia with the following positions.
  - a) 15° 25' S, 28° 16' E
  - b) 13° 38' S, 32° 28' E
  - c) 15° 30' S, 25° 30' E
  - d) 16° 30' S, 28° 50' E

2 Work with a world atlas. The position of a point on a map is given as 30° S, 27° E.
  - a) On which side of the Equator (above or below) would you look for the point 30° S, 27° E on a map of the world?
  - b) On which side of the 0° line of longitude (right or left) would you find this point?
  - c) Use your atlas to find the country where we find this point. Name the country.
  - d) Which town lies closest to this point?

3 Use an atlas to find the cities with the following coordinates:
  - a) 60° N, $10\frac{1}{2}^\circ$ E
  - b) 35° N, 140° E
  - c) 20° N, 100° W
  - d) 35° S, 59° W

4 Give the coordinates of the cities as accurately as you can:
  - a) Djibouti (Djibouti)
  - b) Bern (Switzerland)
  - c) Colombo (Sri Lanka)
  - d) Caracas (Venezuela)
  - e) Harare (Zimbabwe)
  - f) Sydney (Australia)

5 Determine the following without looking at an atlas:
  - a) Which of the cities in Questions 3 and 4 are in the Southern Hemisphere?
  - b) Which city from Questions 3 and 4 is closest to the Equator?

Using latitude and longitude to calculate distances

Lines of longitude and latitude are not only used to find position. They are also used to calculate distances in Earth geometry. Fig. 6.15 shows how the latitude of a place on Earth corresponds to an angle measured from the centre of the Earth.

For example, any point on the small circle of latitude 30° N forms an angle of 30° with the radius drawn at the Equator.

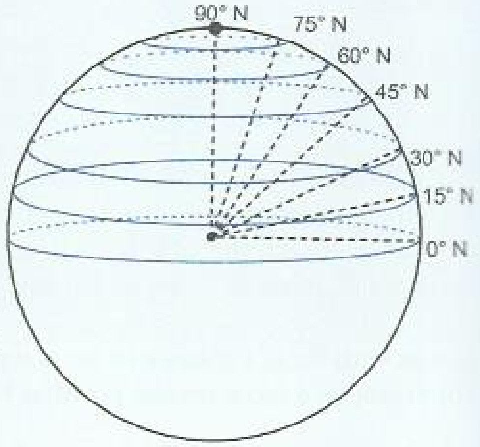

Figure 6.15

Angular di

Angular distan
For your intere
- One degree c
distance.
- One minute

Below we devel
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Calculating
latitude

In Fig. 6.16, A a
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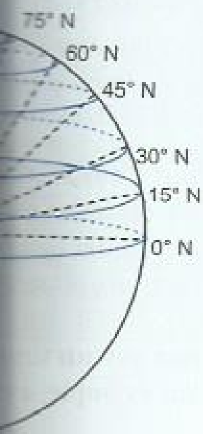

Angular distance

Angular distance is the difference in latitude or longitude between two places. For your interest, picture a globe of the Earth and understand that:

- One degree of angular distance gives an arc of approximately 110.9 km linear distance.
- One minute of angular distance is equal to about 1.83 km of linear distance.

Below we develop a formula for using the difference in latitude or longitude to find the distance between two places. Remember that measuring distance in two dimensions, as on a flat map, would be inaccurate due to the curvature of the Earth's surface.

Calculating the length of an arc on a line of longitude or latitude

In Fig. 6.16, A and B lie on the same line of longitude. M and N lie on the same line of latitude. We can derive formulae for the length of an arc from our circle formulae on page 147.

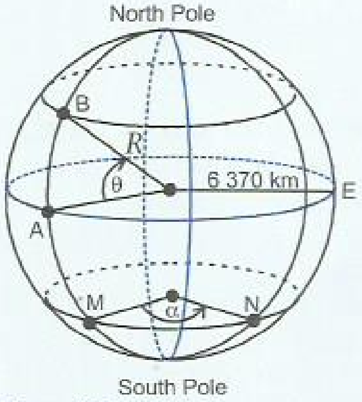

Figure 6.16

To calculate the length of arc AB which lies on a line of longitude (a great circle), we use the formula:

$$AB = \frac{\theta}{360^\circ} \times 2\pi R$$

$$\therefore AB = \frac{\theta}{180^\circ} \times \pi R$$

where θ is the difference in latitude between A and B, and R (= 6 370 km) is the radius of the Earth.

To calculate the length of arc MN which lies on a line of latitude (a small circle), we use the formula:

$$MN = \frac{\alpha}{360^\circ} \times 2\pi r$$

$$\therefore MN = \frac{\alpha}{180^\circ} \times \pi r$$

where α is the difference in longitude between M and N, and r is the radius of the small circle of latitude through M and N.

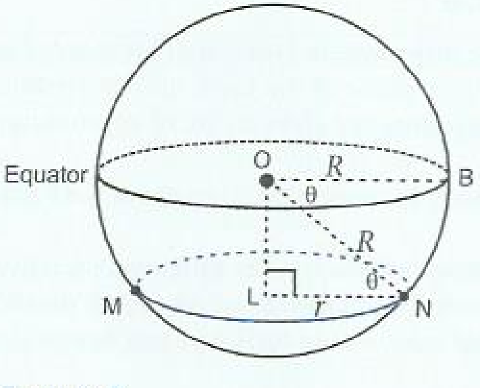

Figure 6.17

Let the radius of the Earth be R = OB = ON.

Let the radius of the small circle be r = LN.

Let the angle between OB and ON be θ. In other words, the small circle is θ degrees south of the Equator.

So in right-angled triangle ΔOLN, ∠ONL = θ, since OB is parallel to LN.

So $\cos\theta = \frac{LN}{ON} = \frac{r}{R}$ $\quad$ $(\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}})$

$\therefore r = R\cos\theta$

The formula for calculating r from R, given the angle θ formed between the radius from the Equator and the small circle, is:

$$r = R\cos\theta$$

Worked example 3

1 A small circle is at a position so that it forms an angle of 55° with the radius at the Equator. Find the radius of the small circle.

2 A circle of latitude has a radius of 3 185 km. Calculate the angle that this circle makes with the radius from the equator.

Note

Because a small circle of the Earth is a line of latitude, we can also call it a circle of latitude or a parallel of latitude.

Worked ex

Answers

1 Draw a ro
The radius

In ΔOLN,
$\therefore r = R\cos$
$= 6\ 37$
$= 3\ 65$
The radius
is 3 654 kr

2 Draw a rou
Here $r = 3$
circle mak
Equator be
$r = R\cos\theta$
$\cos\theta = \frac{r}{R}$
$= \frac{3.18}{63}$
$= 0.5$
$\theta = \cos^{-1} 0.$
$= 60°$
The requir

Activity 4

1 Calculate t
of latitude.
a) 65° S
b) 27° N
c) 82° S
d) 16° N
2 Zambia has
of the lengt
3 Find the an
a) radius 2

Worked example 3 (continued)

Answers

1 Draw a rough sketch of the situation. See Fig. 6.18.
The radius R of the Earth is 6 370 km. We want to find r, the length of LN.

In ΔOLN, $\cos 55° = \frac{r}{R}$

$\therefore r = R\cos 55°$

$= 6\ 370 \times \cos 55°$

$= 3\ 654$

The radius of the circle of latitude 55° N is 3 654 km.

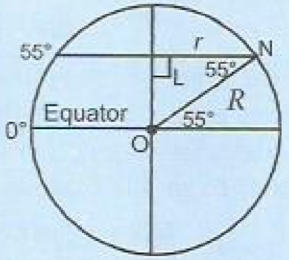

Figure 6.18

2 Draw a rough sketch. See Fig. 6.19.
Here $r = 3\ 185$ km. Let the angle that this circle makes with the radius from the Equator be θ° S.

$r = R\cos\theta$

$\cos\theta = \frac{r}{R}$

$= \frac{3\ 185}{6\ 370}$ (from ΔOLN)

$= 0.5$

$\theta = \cos^{-1} 0.5$

$= 60°$

The required latitude is therefore 60° S.

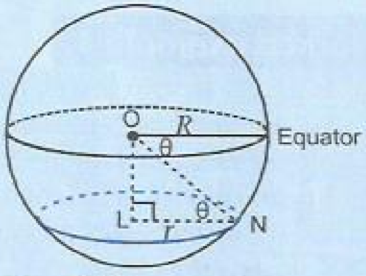

Figure 6.19

Activity 4

1 Calculate the radius of the small circles parallel to the Equator at these lines of latitude.
- a) 65° S
- b) 27° N
- c) 82° S
- d) 16° N

2 Zambia has a range of lines of latitude from 9° S to 18° S. What is the range of the length of the radii of the small circles parallel to the Equator?

3 Find the angle from the Equator that each of these lines of latitude makes:
a) radius 2 445 km   b) circumference 21 400 km

Another way of calculating the length of an arc on a circle of latitude

We have learnt that to calculate the length of the arc MN which lies on a circle of latitude, given the value of the radius r of the small circle, we use the formula:

$$MN = \frac{\alpha}{180°} \times \pi r$$

However, if are not given the value of r but are given the angle θ between the small circle and the Equator (see Fig. 6.2) we can substitute $r = R\cos\theta$ to get this formula for the length of MN:

$$MN = \frac{\alpha}{180°}\pi R\cos\theta$$

where θ is the angle between the radius of the small circle and R, the radius of the Earth.

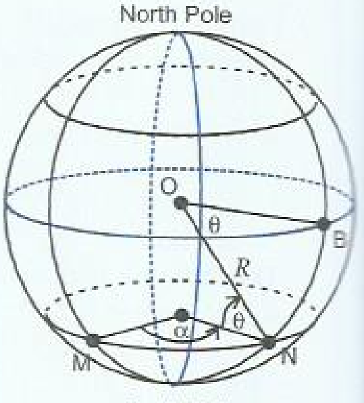

Figure 6.20

Worked example 4

1 M and N are two places, both lying on latitude 30° north. M is on longitude 85° 15' west and N is on longitude 30° 30' west. Calculate
   - a) the radius of latitude 30° north
   - b) the difference in longitude between M and N
   - c) the distance MN measured along the surface of the earth.

2 Calculate the distance CD, given that C is on latitude 60° south of the Equator and D is on latitude 60° north of the Equator.

Answers

1 Fig. 6.21 shows the relative positions of M and N (not to scale).

a) $r = R\cos 30°$

$= 6\,370 \times 0.8660254\ldots$

$= 5\,517$

The radius of latitude 30° is therefore 5 517 km

b) Difference in longitude between M and N = ∠MTN

∠MTN = 85° 15' – 30° 30' = 54° 45'.

This gives us the value of α, the angle between the radii of the small circles.

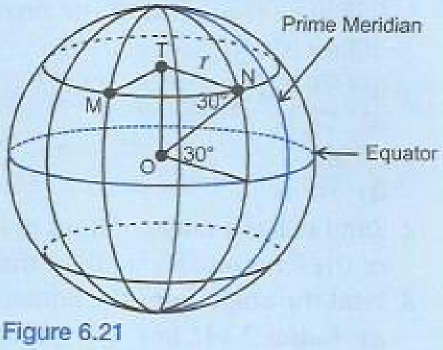

Figure 6.21

c) Length of arc MN $= \frac{\alpha}{180°}\pi r$

$= \frac{54°\,45'}{180°} \times 3.142 \times 5\,517$

$= 5\,273$

$\therefore$ MN = 5 273 km.

Worked exa

2 C and D are
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Length of ar

$= \frac{120°}{180°} \times \pi \times$

= 13 343

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Activity 5

1 Copy the gri
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and longitud
   - a) A(0° E, 0°
   - b) B(30° E, 6
   - c) C(0° E, 30
   - d) D(30° W,
   - e) E(60° W, 3
   - f) F(90° W, 3
   - g) G(30° E, 9
   - h) H(15° W,

2 Give the latitu

3 Fig. 6.24 show
that follow.
   - a) Write dow
in terms o
longitude
A to F.
   - b) Find the di
latitude be
      - i) A and
      - iii) C and
      - v) G and
      - vii) B and
      - ix) K and

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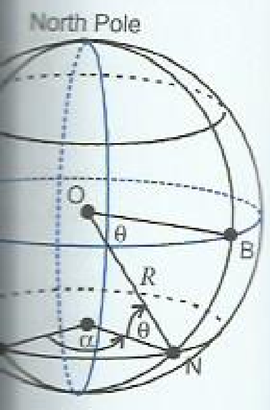

South Pole

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M is on longitude

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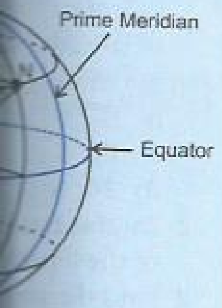

Worked example (continued)

2 C and D are points on the same longitude, therefore they lie on a great circle. We need to find the length of the arc CD. C and D are on opposite sides of the Equator, therefore $\angle COD = 60° + 60° = 120°$.

$\text{Length of arc } CD = \frac{\theta}{180°} \times \pi R$

$= \frac{120°}{180°} \times \pi \times 6\,370$

$= 13\,343$

So the distance CD is 13 343 km.

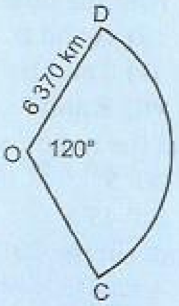

Figure 6.22

Activity 5

1 Copy the grid in Fig. 6.23. Plot in the following latitudes and longitudes.

a) A(0° E, 0° N)
b) B(30° E, 60° N)
c) C(0° E, 30° S)
d) D(30° W, 30° N)
e) E(60° W, 30° S)
f) F(90° W, 30° S)
g) G(30° E, 90° S)
h) H(15° W, 0° S)

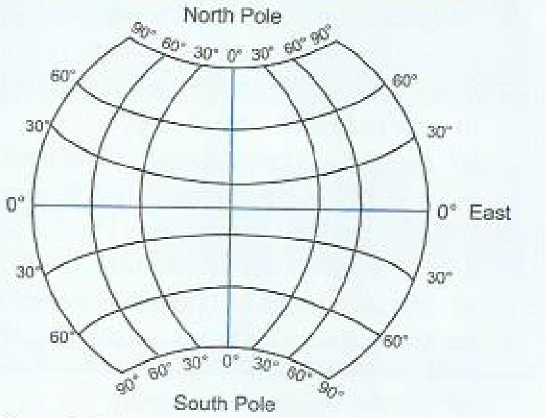

Figure 6.23

2 Give the latitude and longitude of the South Pole and the North Pole.

3 Fig. 6.24 shows part of the Earth's grid. Use the grid to answer the questions that follow.

a) Write down the positions in terms of latitude and longitude of the points A to F.

b) Find the difference in latitude between:

i) A and C	ii) A and I
iii) C and I	iv) I and L
v) G and M	vi) N and F
vii) B and F	viii) K and F
ix) K and N	

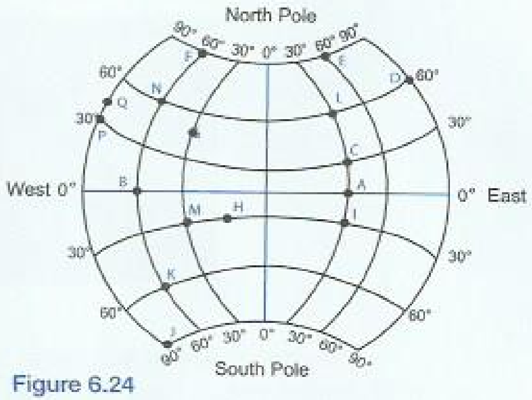

Figure 6.24

Activity 5 (continued)

c) Find the difference in longitude between:

i) A and B ii) D and L iii) D and N
iv) I and H v) I and M vi) H and M
vii) E and F viii) C and P ix) Q and G

4 Find the radius of each of the following circles of latitude.

a) 60° S b) 45° N c) 33.4° N d) 0°
e) 80° 45' S f) 83° 53' S g) 90° N h) $66\frac{2}{3}°$ N

5 Work out the distance between the following places on the grid in Question 2.

a) A and C b) A and I
c) C and I d) I and L
e) G and M f) A and B
g) D and L h) D and N
i) I and H j) I and M

6 Calculate the distance between the North Pole and the South Pole, measured along the Greenwich Meridian.

7 Town A is (47° N 56° E) and town B is (47° N 56° W), find:

a) the radius of latitude 56°
b) the difference in longitude, between latitude 56° E and latitude 56° W
c) the distance between A and B, measured along latitude 56°.

8 C and D are two towns lying on the same longitude. If C lies on latitude 46° 36' S and D lies on latitude 52° 28' N, calculate:

a) the difference in latitude between C and D
b) the distance CD measured along the longitude through C and D.

Nautical

Nautical m

Distances at sea
as a sea mile.

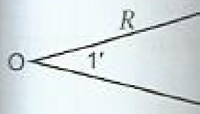

Figure 6.25

In Fig. 6.25, ΔA
S = 1 nmi where
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The knot

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Activity 6

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SUB-TOPIC 4 Speed in knots and time

Nautical miles and knots

Nautical mile

Distances at sea are measured in nautical miles, so we can think of a nautical mile as a sea mile.

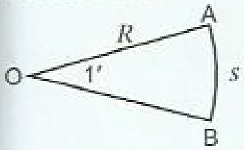

Figure 6.25

New word

nautical mile (nmi): unit of distance that is approximately one minute of arc measured along any meridian of the Earth.

In Fig. 6.25, ΔAOB = 1' or $\frac{1}{60}°$ therefore $S = 1$ nmi where S is the length of the arc AB and R is the radius of the Earth. 1 nmi = 1.853 km

Note

$$S = \frac{\left(\frac{1}{60}\right)°}{360°} \times 2 \times 3.142 \times 6\,370 \text{ km} = 1.853 \text{ km}$$

The knot

The knot is associated with speed of vessels at sea, wind speed and speed of aircraft.

New word

knot (kn): unit of speed equal to one nautical mile per hour

Activity 6

1. Two places P and Q lie on the same longitude and are 30° apart in latitude, find the distance between them in: **a)** kilometres **b)** nautical miles
2. Two places P and Q lie on the Equator and are 5' apart, calculate the distance PQ in: **a)** kilometres **b)** nautical miles
3 If a train is moving at 100 km/h on a track, how fast is this in knots?
4 The Cape-to-Rio yacht race follows the course shown in Figure 6.26.

Figure 6.26

a) Calculate the distance in nautical miles and in kilometres.
b) The fastest recorded time for the race is 12 days and 16 hours. What was the average speed in **(i)** knots **(ii)** kilometres per hour.

Did you know?

The new time standard for the world is called **Coordinated Universal Time (UTC)** (from the French: *temps universel coordonné*). This has replaced Greenwich Mean Time (GMT), and is more scientifically defined. It is independent of the time in Britain, which changes between summer and winter. However, we use GMT to mean the time denoted by the line of 0° longitude.

Time is measured from the Greenwich Meridian. Time measured from this meridian is known as **Greenwich Mean Time** (GMT) and is universal. This is universal time in that time throughout the world is determined from this longitude. The earth rotates once on its axis every 24 hours. This means that every 24 hours the earth describes an angle of 360° about its axis.

So 24 hours = 360°

$\therefore$ 1 hour = $\frac{360°}{24} = 15°$

The earth rotates from west to east. Since 1 hour is equivalent to 15°:

- for every 15° travelled eastwards an hour is gained
- for every 15° travelled westwards an hour is lost.

However, every country chooses its own time zone, or time zones, as shown in the time zone map in Fig. 6.27.

Zambian time is determined from longitude 30° E. If we divide this by 15° we get 2. This is why Zambian time is 2 hours ahead of Greenwich Mean Time. So when we hear the TV or radio say that the time is 14:00 GMT then it is 16:00 Zambian time.

Note

The simple rule for determining time between longitudes is: "going east add and going west subtract".

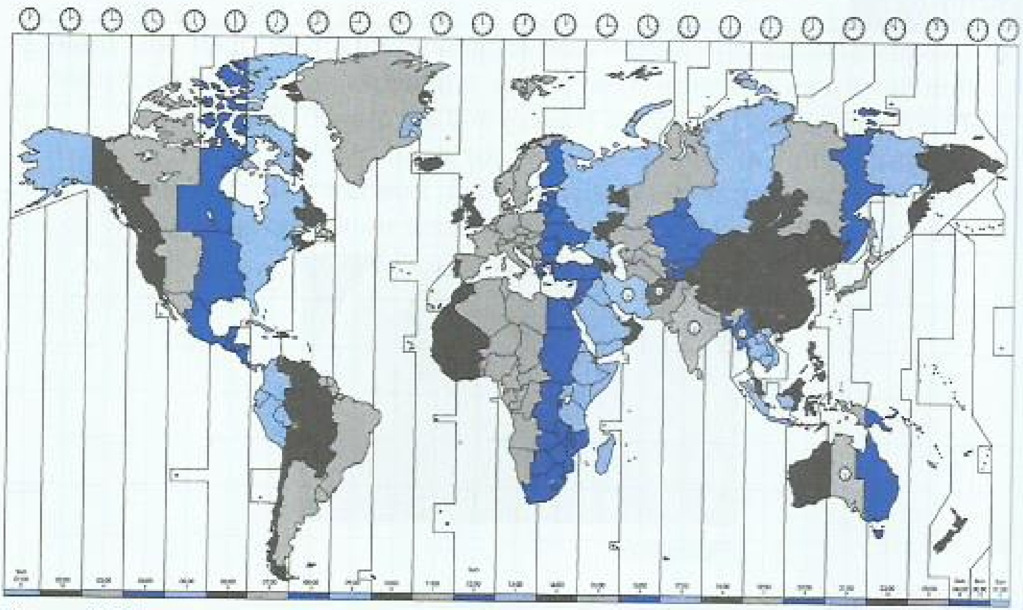

Figure 6.27

Worked e

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Answers

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b) $\frac{95°}{15°}$ =

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Activity 7

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Worked example 6

1 If the time is 10:00 GMT, calculate the local time in the following places.

a) Madras on longitude 80° east b) Galveston, Texas, US 95° west

2 Town A is on longitude 30° west and town B is on longitude 60° east.
If it is 12:00 universal time at B, what is the time at A?

Answers

1 a) $\frac{80^\circ}{15^\circ} = 5\frac{1}{3}$, this means there is a difference of $5\frac{1}{3}$ hours between Greenwich and Madras. $5\frac{1}{3}$ = 05 h 20'
The time in Madras is therefore 10:00 + 05 h 20' = 15:20.

b) $\frac{95^\circ}{15^\circ}$ = 06 h 20'. The time in Galveston is therefore 10:00 – 06 h 20' = 03 h 40'

2 Difference in longitude between the towns is 30° + 60° = 90°.
Difference in time between the towns is $\frac{90^\circ}{15^\circ}$ = 6 hours
Since A is west of B, the time at A is 6 hours behind the time at B.
Therefore time at A is 12 – 6 = 06:00.

Activity 7

Consider all distances between places to be measured along the minor arc.

1 If the time is 15:00 GMT find the time at each of the following.

a) P 45° east b) Q 90° east c) R 120° west
d) S 144° west e) Lisbon 8° west f) Candala 50° east
g) Yehsien 120° east h) Idaho 112° west.

2 Find the difference in time between the following longitudes:

a) 50° east and 10° west b) 20° east and 50° east
c) 10° east and 170° west d) 20° 15' east and 50° 45' east
e) 25° 25' west and 50° 50' east f) 45' west and 25° 25' east

3 a) Explain why the International Date Line has this name.

b) The earth rotates from west to east. This means that the sun rises in the east. Which continent and which country will be the first to start a new day? Which island was the first to celebrate the new millennium in 2000?

4 a) How many time zones does Australia have? Explain your answer.

b) If it is midday (12 o'clock) in London, what time will it be in Perth on the west coast of Australia? Explain your answer.

5 The US has four main time zones which it shares with Canada (Eastern Time, Central Time, Mountain Time and Pacific Time).

a) If it is 20:00 in the UK, what is the time in Central Time in the US?

b) If you travelled from a place which has Eastern Time to a place which has Mountain Time, how would you need to adjust the time on your watch?

c) If you travelled from a place which has Pacific Time to a place which has Central Time, how would you need to adjust your watch?

d) If you travelled to the east coast of the US from Zambia, how would you need to adjust your watch?

TOPIC
6 Summary, revision and assessment

Summary

- Euclidean geometry is done on a flat plane, while Earth geometry takes the Earth's curvature into account.
- We use Earth geometry for locating places on the Earth's surface and determining distances and universal time between such places.
- Lines of latitude are imaginary parallel lines that run from east to west around the Earth's surface. The longest of these is called the Equator and is the only line of latitude with a radius of 6 370 km.
- Lines of longitude (meridians) represent east–west location.
- A great circle of the Earth is a circle on the surface of the Earth whose radius is equal to that of the Earth. The Equator and all lines of longitude are great circles and a great circle can be drawn at any point on the Earth's surface.
- A small circle of the Earth is a circle on the surface of the Earth whose radius is less than the radius of the Earth. All lines of latitude, except the Equator, are small circles. There are other small circles that are not latitudes.
- The shortest surface distance between any two points on a sphere is the length of the arc along the great circle through those points.
- A formula for the length of an arc AB on a line of longitude is:

  $$AB = \frac{\theta}{360^\circ} \times 2\pi R \text{ simplified to } \frac{\theta}{180^\circ}\pi R$$

  where θ is the difference in latitude between A and B, and R (= 6 370 km) is the radius of the earth.
- A formula for the length of an arc MN on a line of latitude is:

  $$MN = \frac{\alpha}{360^\circ} \times 2\pi r, \text{ simplified to } \frac{\alpha}{180^\circ} \times \pi r$$

  where α is the difference in longitude between M and N, and r is the radius of the small circle of latitude through M and N.
- A nautical mile (nmi) is a unit of distance that is approximately one minute of arc measured along any meridian of the Earth.
- A knot (kn) is a unit of speed equal to one nautical mile per hour
- Time at different parts of the Earth differs depending on the longitude.
- Times are measured from the Greenwich Meridian, 1 hour $= \frac{360^\circ}{24} = 15^\circ$
- Times can be calculated by using the number of degrees longitude west or east of the Greenwich Meridian. The simple rule for determining time between longitudes is: "going east add and going west subtract".
- Every country chooses its own convenient time zones. Zambian time is 2 hours ahead (to the east) of GMT. So when we hear the TV or radio say that the time is 14:00 GMT then it is 16:00 Zambian time.

Summary, re

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7 China covers
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Summary, revision and assessment (continued)

Revision exercise

1 A and B are two positions on the Equator. Given that A lies on longitude 30° west of the Greenwich Meridian and B lies on longitude 60° east of the Greenwich Meridian, calculate the distance between A and B.

2 Lusaka L is (28° 30′ E, 15° 30′ S) and Bulawayo B is (28° 30′ E, 21° S). Taking the radius of the earth as 6 370 km, calculate the distance between the two cities as measured along longitude 28° 30′ E.

3 Lusaka is (28° 30′ E, 15° 30′ S) and Cuiaba in Brazil is (56° 05′ W, 15° 30′ S), find:
  - **a)** the difference in longitude between Lusaka and Cuiaba
  - **b)** the distance between Lusaka and Cuiaba.

4 P is a landmark on the South Pole and Q is another landmark on latitude 28° north and longitude 30° west. Calculate:
  - **a)** the difference in latitude between P and Q
  - **b)** the distance PQ measured along longitude 30° west.

5 Ankara is (34° E, 40° N) and Beijing is (117° E, 40° N). Calculate:
  - **a)** the difference in longitude between Ankara and Beijing
  - **b)** the radius of latitude 40°
  - **c)** the distance between Ankara and Beijing measured along latitude 40° N.

6 Macapa in Brazil and Libreville in Gabon lie on the Equator. Macapa is on longitude 52° west and Libreville is on longitude 10° east. A plane leaves Macapa for Libreville at 13:45 and travels at 500 km/h, find:
  - **a)** the distance between Macapa and Libreville
  - **b)** the time to the nearest hour taken by the plane on the trip.
  - **c)** the time in Libreville when the plane arrives.

7 China covers a span of sixty degrees longitude, but has only one standard time zone.
  - **a)** Find the lines of longitude on the map and write down their references.
  - **b)** If the sun rises at 5.30 a.m. in the most eastern part of China, at what time does it rise in the most western part of China?
  - **c)** If the sun sets at 7 p.m. in the most western part of China, at what time does it set in the most eastern part of China?

Summary, revision and assessment (continued)

*8 **[Extension]** The Antarctic Circle has latitude 66° S.

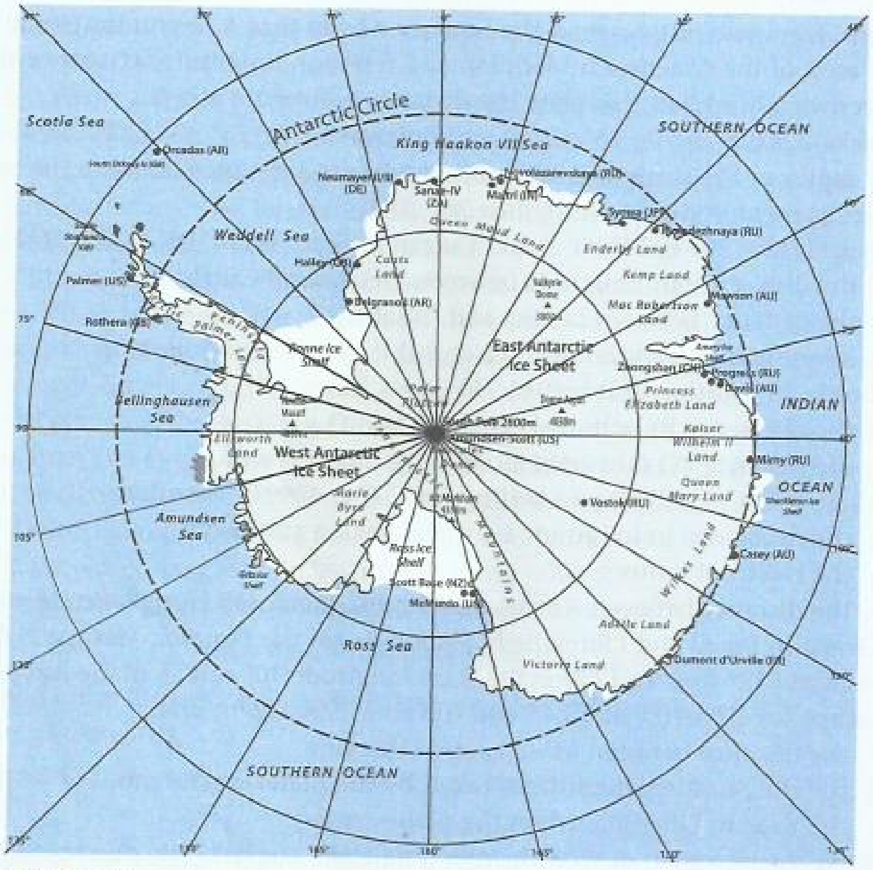

Figure 6.28

a) Calculate the distance from Dumont d'Urville to Mirny in nautical miles.

b) If a small aeroplane flies from Dumont d'Urville to Mirny at an average speed of 90 knots, how long will it take?

Assessment

1 Town A has co
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2 In Fig. 6.29, A
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Assessment exercise

1 Town A has coordinates 46° E, 60° N and Town B x° W, 60° N. Given that A and B are on the opposite sides of the Earth, calculate:
   a) the value of x
   b) the radius of latitude 60°
   c) the distance between the towns measured along latitude 60°.
   d) the distance between the towns measured. along the great circle through the North Pole.

2 In Fig. 6.29, A and B are towns on latitude 34° N with centre C. O is the centre of latitude 0° and N and S are the North and South Poles. Given that ACB is a straight line, calculate:
   a) AC, the radius of latitude 34°
   b) the length of the arc ANB
   c) the distance AB measured along latitude 34°
   d) the length of OC
   e) the length of CN.

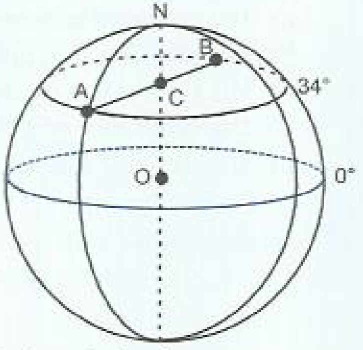
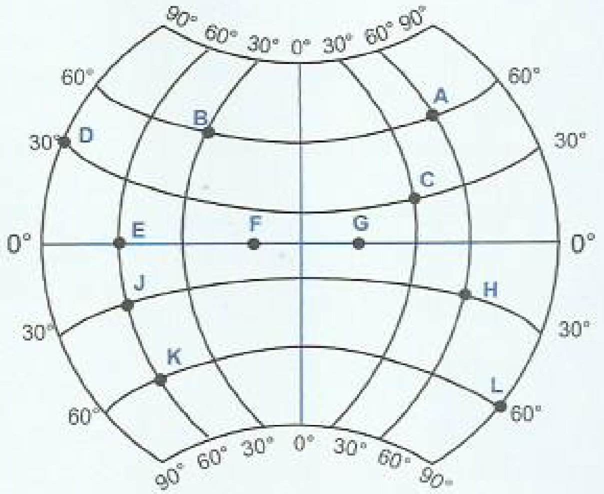


Figure 6.29

3 For the grid shown below, find:



   a) the time at A if it is 04:00 hours at B
   b) the difference in time between C and D
   c) the time at C if it is 15:42 at D
   d) the difference in time between E and F
   e) the time at E when it is 05:25 at F
   f) the difference in time between G and F
   g) the time at G when it is 00:36 at F

Summary, revision and assessment (continued)

- h) the time difference between J and H
- i) the time at J when it is 17:55 at H
- j) the time difference between E and K
- k) the time at K when it is 17:55 at E.

4. On a Monday at 06:40 a plane leaves a stationary aircraft carrier A on (20° S, 30° W), flies east at 450 knots and lands on another stationary carrier B on (20° S, 90° E), find:
   - a) the radius of latitude 20° south
   - b) the distance in nautical miles between the carriers
   - c) the time taken by the plane in flying from A to B
   - d) the difference in time between A and B
   - e) the day and time at B when the plane lands.

TOPIC
7

Sub-
Differentiation
Integration

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TOPIC 7

Introduction to calculus

Sub-topic	Specific Outcomes
Differentiation	• Explain the concept of differentiation. • Differentiate functions from first principles. • Use the formula for differentiation. • Calculate equations of tangents and normal.
Integration	• Explain integration. • Find indefinite integrals. • Evaluate simple definite integrals. • Find the area under a curve.

Calculus is the basis of advanced Mathematics, as it deals with changing situations. The two main concepts are differentiation and integration.

Starter activity

Scientists believe that a certain type of bacteria grows according to the rule $n = t^2 - 6t + 10$, where n is the number of bacteria in millions and t is the time in seconds.

Fig. 7.1 shows the graph of the function $n = t^2 - 6t + 10$. Answer the questions below based on this graph.

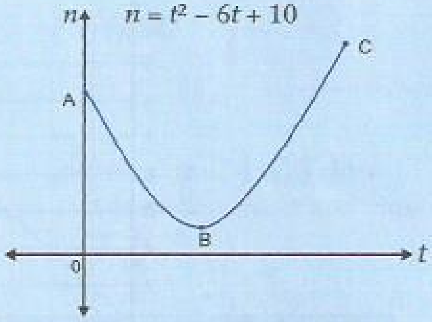

Figure 7.1

1 Use the graph and the equation to find the smallest number of bacteria during the time period shown.
2 Find the rate at which the bacteria are increasing during the period from $t = 2$ to $t = 5$.
3 What can you say about the portions AB and BC of the graph?
4 Do you see a place on the graph where the rate of growth is neither increasing nor decreasing? Explain how you know this.
5 Remember what you learnt in Topic 1 about stationary points. Show algebraically that the point in Question 4 is a stationary point.

SUB-TOPIC 1 Differentiation

In Topic 1, you learnt how to apply differentiation to find stationary points and the point of inflection of cubic functions. When you differentiate a function you are finding another function that describes the rate of change, or slope, of the first function.

We are going to study the principles of differentiation in more detail in this topic.

The concept of gradient

Consider the following example. A learner observes the growth of a tree sapling from a seedling. She records the results as shown in the table.

Time (t weeks)	0	1	2	3	4
Height (h cm)	2	4	6	8	10

Fig. 7.2 depicts a graph of the information.

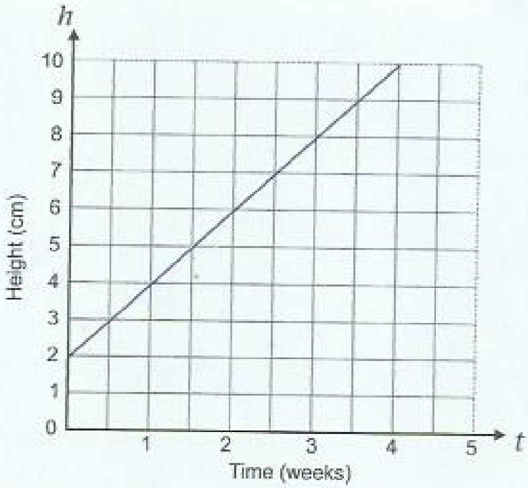

Figure 7.2

Note that the gradient of height against time is a straight line intersecting the vertical axis at the point (0, 2). This means that the observation started at the time when the plant was 2 cm tall. The gradient of the line represents the rate per week at which the plant was growing.

$$\text{Gradient of the line} = \frac{\text{change in height}}{\text{time taken}}$$
$$= \frac{10 - 2}{4}$$
$$= 2 \text{ cm/week}$$

So the plant was growing at a constant rate of 2 cm per week.

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Gradient of a curve

The gradient of a curve constantly changes. The gradient of a curve at a point is the gradient of the tangent to the curve at the point. Consider finding the gradient of the curve $y = x^2$ at the point P(1, 1).

Let A be a variable point on the curve $y = x^2$. Move point A towards point P to take positions A_1, A_2, A_3, and so on.

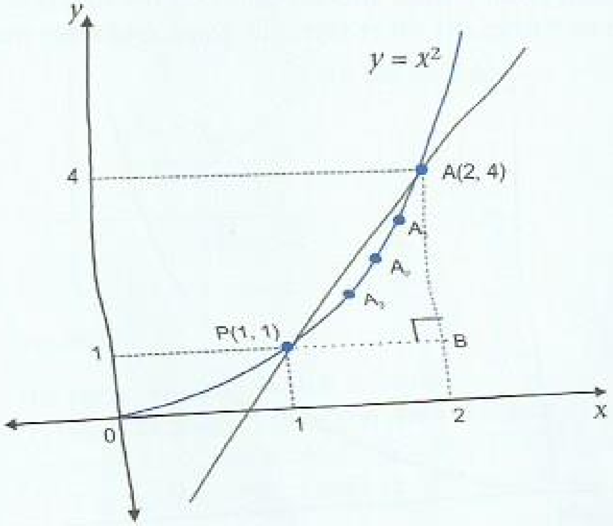

Figure 7.3

Now find the gradient of line PA in each of these positions until A is as close as possible to P. The table below shows the results as A is moved closer and closer to P.

In the initial position the gradient of PA is given by $\frac{AB}{PB} = \frac{4-1}{2-1} = 3$

Coordinates of A	AB	PB	Gradient of PA = $\frac{AB}{PB}$
(2, 4)	4 − 1 = 3	2 − 1 = 1	$\frac{3}{1} = 3$
(1.5, 2.25)	1.25	0.5	2.5
(1.4, 1.96)	0.96	0.4	2.4
(1.3, 1.69)	0.69	0.3	2.3
(1.2, 1.44)	0.44	0.2	2.2
(1.1, 1.21)	0.21	0.1	2.1
(1.05, 1.1025)	0.1025	0.05	2.05
(1.01, 1.0201)	0.0201	0.01	2.01

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Gradient of a curve

The gradient of a curve constantly changes. The gradient of a curve at a point is the gradient of the tangent to the curve at the point. Consider finding the gradient of the curve $y = x^2$ at the point P(1, 1).

Let A be a variable point on the curve $y = x^2$. Move point A towards point P to take positions A_1, A_2, A_3, and so on.

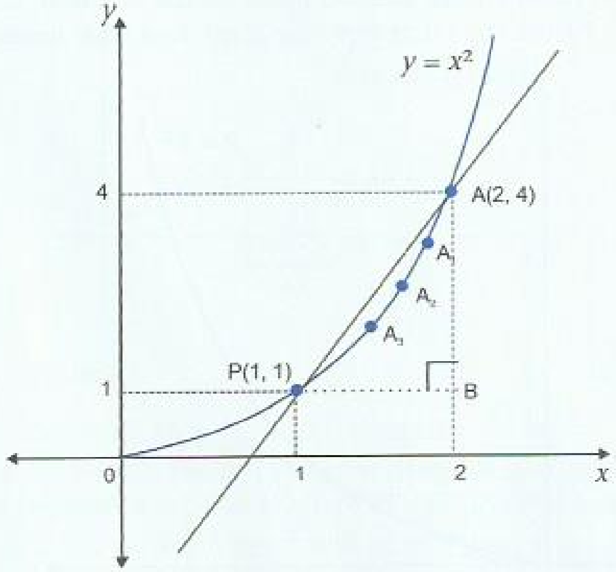

Figure 7.3

Now find the gradient of line PA in each of these positions until A is as close as possible to P. The table below shows the results as A is moved closer and closer to P.

In the initial position the gradient of PA is given by $\frac{AB}{PB} = \frac{4-1}{2-1} = 3$

Coordinates of A	AB	PB	Gradient of PA = $\frac{AB}{PB}$
(2, 4)	4 − 1 = 3	2 − 1 = 1	$\frac{3}{1} = 3$
(1.5, 2.25)	1.25	0.5	2.5
(1.4, 1.96)	0.96	0.4	2.4
(1.3, 1.69)	0.69	0.3	2.3
(1.2, 1.44)	0.44	0.2	2.2
(1.1, 1.21)	0.21	0.1	2.1
(1.05, 1.1025)	0.1025	0.05	2.05
(1.01, 1.0201)	0.0201	0.01	2.01

Notice that as A gets closer and closer to P, the gradient of PA gets closer and closer to 2. It never reaches 2, because A cannot be at the same place as P, otherwise the ratio $\frac{AB}{PB}$ would be impossible to find.

So we can say, as point A approaches point P, straight line PA approaches the tangent at P, and its gradient approaches 2.

Although we cannot find the gradient $\frac{AB}{PB}$ at PA = 0, we can get as close as we like to this value. This leads us to the concept of a **limit**.

If we approached point P from another point on the function, C, which gets closer and closer to P from the other side, our graph and table would look like this:

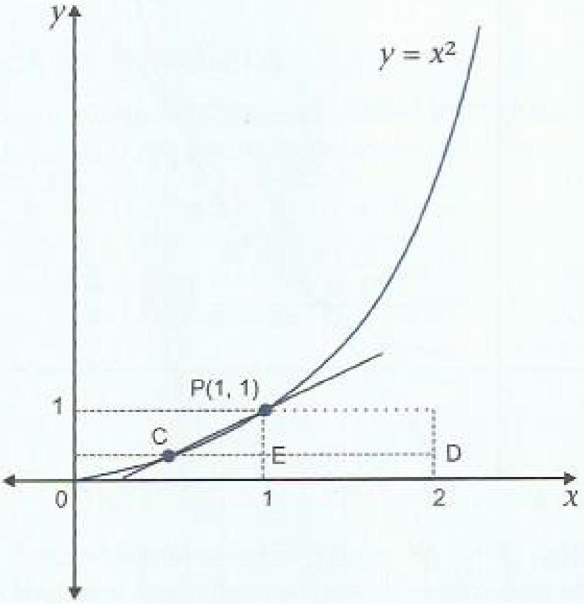

Figure 7.4

Coordinates of C	PE	EC	Gradient of PC = $\frac{PE}{CE}$
(0.5, 0.25)	0.75	0.5	$\frac{0.75}{0.5} = 1.5$
(0.8, 0.64)	0.36	0.2	1.8
(0.9, 0.81)	0.19	0.1	1.9
(0.95, 0.9025)	0.0975	0.05	1.95
(0.97, 0.9409)	0.0591	0.03	1.97
(0.98, 0.9604)	0.0396	0.02	1.98
(0.99, 0.9801)	0.0199	0.01	1.99

gets closer and closer
e as P, otherwise the

PA approaches the

t as close as we like

ion, C, which gets
would look like

dient of PC = $\frac{PE}{CE}$
$\frac{0.75}{0.5} = 1.5$
1.8
1.9
1.95
1.97
1.98
1.99

Differentiation from first principles

We generalise the situation by working out a formula for the gradient of a curve at any point.

In order for a function to be differentiable at an x-value of a, a limit must exist at a. Look at the graph of function $f(x)$ in Fig. 7.5.

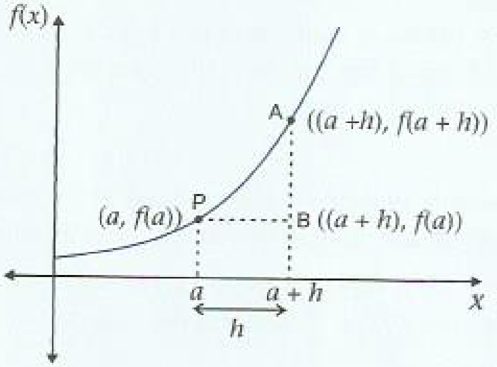

Figure 7.5

The gradient of the chord PA is $\frac{AB}{PB}$, which is equal to $\frac{f(a+h)-f(a)}{h}$.

As point A gets closer to point P, then the chord PA gets closer to the tangent at point P. As this happens $h \to 0$. The gradient of a graph f at a point $(x, f(x))$ is equal to $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$. The notation $\lim_{h \to 0}$ is read as "the limit as h tends to zero".

The process of finding the gradient of a curve by using small increases, which we call h, is known as **differentiation from first principles**.

The formula for differentiation by first principles is:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

If you apply this formula to a function, you will see that it gives the same result as the rules for differentiation you learnt in Topic 1.

Calculating limits

We need to know how to calculate limits. These can be used for ordinary functions. If you know the value of the function at the value of the limit, then that is the limit.

So the first thing to try is to substitute the value into the function. For example, if we have $f(x) = \frac{x}{x+4}$, and we need to find $\lim_{x \to 1} f(x)$ then we can take the value of this function at $x = 1$.

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x}{x+4} = \frac{1}{1+4} = \frac{1}{5}$$

This function becomes very close to $\frac{1}{5}$ when x gets very close to 1.

A limit is also useful if a function is undefined at a point, as in the case of our gradient at a point. In this case, we find the limit by finding the value that the function approaches as it gets closer and closer to that point.

The limit of a constant function is simply the value of that constant. For example, if $f(x) = 4$ then $\lim\limits_{x \to 1} f(x) = 4$.

Here are some rules for calculating limits.

- For a function where the function value exists for that value of x, substitute the value of x to get the limit.
- For a function where the function value does not exist for that value of x, you will need to change the form of the function into one for which it is defined, as you will see in the next worked example.

Alternative notation

An alternative notation used to determine the gradient of a curve $y = f(x)$ at any point $P(x, y)$ is $\frac{dy}{dx}$, where dy means change in y and dx means change in x.
So the gradient of the curve $y = f(x)$ at any point $P(x; y) = \frac{dy}{dx}$.

If $y = f(x)$, then the notation $f'(x) = \frac{dy}{dx}$ denotes the first derivative of y with respect to x.
$f'(2)$ gives the gradient of the curve $y = f(x)$ at the point where $x = 2$.
$f'(0)$ gives the gradient of the curve $y = f(x)$ at the point where $x = 0$, and so on.

Worked example 1

1 Given the function $g(x) = \frac{x^2 - 9}{x - 3}$, calculate $\lim\limits_{x \to 3} \frac{x^2 - 9}{x - 3}$.

2 Find from first principles the gradient of the curve $f(x) = 2x^2 + x$ at the point (2, 10).

3 Find from first principles the gradient of the curve $s(x) = 2x^3 - 3x + 2$ at the point where $x = 4$.

Answers

1 We want the limit as $x \to 3$, but we can't evaluate the function at $x = 3$. This is because the denominator equals $x - 3$, so for $x = 3$ the denominator would be 0, so the function would be undefined. We have to change the form of the function:

$\lim\limits_{x \to 3} \frac{\cancel{(x - 3)}(x + 3)}{\cancel{x - 3}}$ (Factorise the numerator and cancel the common terms.)

$= \lim\limits_{x \to 3} (x + 3)$ (State the restriction: $x \neq 3$)

$\therefore \lim\limits_{x \to 3} (x + 3) = 6$

We cannot let $x = 3$, but we can make it as close to 3 as we like.
So as x tends to 3, the limit of the function is 6.

2 We need to use first principles to find $f'(x)$, which means we apply the formula:

$$f'(x) = \lim_{h \to 2} \frac{f(x + h) - f(x)}{h} \text{ to } f(x) = 2x^2 + x$$

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change in x.

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$2x^3 - 3x + 2$ at the

tion at $x = 3$. This
denominator would
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e common terms.)

e like.

we apply the

Worked example 1 (continued)

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{2(x+h)^2 + (x+h) - (2x^2+x)}{h} \\ &= \lim_{h\to 0}\frac{2(x^2+2xh+h^2)+x+h-2x^2-x}{h} \\ &= \lim_{h\to 0}\frac{\cancel{2x^2}+4xh+2h^2\cancel{+x}+h\cancel{-2x^2}\cancel{-x}}{h} \\ &= \lim_{h\to 0}\frac{4xh+2h^2+h}{h} \\ &= \lim_{h\to 0}(4x+2h+1) \\ &= 4x+1 \end{aligned}$$

At $x = 2$, $f'(2) = 4(2) + 1 = 9$

$\therefore$ the gradient of the curve at (2, 10) is 9.

3 Apply $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ to $s(x) = 2x^3 - 3x + 2$

$$\begin{aligned} &= \lim_{h\to 0}\frac{2(x+h)^3 - 3(x+h) + 2 - (2x^3+3x+2)}{h} \\ &= \lim_{h\to 0}\frac{2(x^3+2x^2h+xh^2+x^2h+2xh^2+h^3)-3x-3h+2-2x^3+3x-2}{h} \\ &= \lim_{h\to 0}\frac{\cancel{2x^3}+6x^2h+6xh^2+2h^3\cancel{-3x}-3h\cancel{+2}\cancel{-2x^3}\cancel{+3x}\cancel{-2}}{h} \\ &= \lim_{h\to 0}\frac{6x^2h+6xh^2+2h^3-3h}{h} \\ &= \lim_{h\to 0}(6x^2+6xh+2h^2-3) \\ &= 6x^2-3 \end{aligned}$$

At $x = 4$, the gradient $f'(x) = 6(4)^2 - 3 = 93$.

Activity 1

1 Calculate the following limits.

a) $\lim_{x\to 1}(2x+5)$
b) $\lim_{x\to 4}(x^2-3x)$
c) $\lim_{x\to 4}(x+1)(2x-1)$
d) $\lim_{x\to 5}16$
e) $\lim_{x\to 2}\frac{x^2+1}{x+2}$
f) $\lim_{x\to 0}\frac{x^3-1}{x-1}$
g) $\lim_{x\to 2}\frac{x-3}{x^2-3x}$
h) $\lim_{x\to -2}\frac{x^2+3x+2}{x+2}$
i) $\lim_{x\to -2}\frac{3x^2+5x+2}{x+1}$

2 Differentiate the following using first principles.

a) $y = 2x + 3$ ✓
b) $y = 3x - 2$
c) $y = 4 - 2x$
d) $y = 2x^2$ ✓
e) $s = 2t^2 - 5$
f) $y = 7x - 2x^2$

3 Find the gradient at the given point by using first principles.

a) $y = 4x + 3$; (1, 7) ✓
b) $y = 2x + 6$, (0, 6) ✓
c) $y = 9 - 2x$; (2, 5)
d) $y = 2x^2 + 1$; (–1, 3)
e) $s = 3t^2 - 7t$; (2, –2)
f) $y = 4x - x^2$; (–1, –5)
g) $y = 2x^2 - 4x + 4$; (0, –4)
h) $v = t^2 + 2t + 3$; (–2, 3)
i) $s = 2 - 2t - t^2$; (1, –1)
j) $s = t - t^2$; (1, 5)

Differentiating using the formula

If $y = ax^n$ where a and n are constants then

$$\frac{dy}{dx} = nax^{n-1}$$

This is the rule that you used in Topic 1 for differentiating any algebraic function of the form $y = f(x)$. This is much faster than using first principles, so we usually use this method (unless a question specifies using first principles).

Worked example 2

Use the differentiation formula to do the following.

1 Differentiate $3x^4$.
2 Differentiate $4x^2 + 5x$.
3 Find $\frac{ds}{dt}$ if $s = 2t^3 - 4t^2 + 6$.

Answers

1 Let $y = 3x^4$

$$\frac{dy}{dx} = 4 \times 3x^{4-1}$$
$$= 12x^3$$

2 Let $y = 4x^2 + 5x$ and then apply the rule to each of the terms.

$$\therefore \frac{dy}{dx} = 2 \times 4x^{2-1} + 1 \times 5x^{1-1}$$
$$= 8x + 5x^0$$
$$= 8x + 5 \qquad (x^0 = 1)$$

3 First write $2t^3 - 4t^2 + 6$ as $2t^3 - 4t^2 + 6t^0$ and then apply the rule to each term.

Therefore $\frac{ds}{dt} = 3 \times 2t^{3-1} - 2 \times 4t^{2-1} + 0 \times 6t^{0-1}$

$$= 6t^2 - 8t + 0$$
$$= 6t^2 - 8t$$

In general, the derivative of a constant is 0. You can do the multiplication and subtraction operations mentally, as in the following example.

Worked example 3

Differentiate $3 + 5x - 2x^3$.

Answer

The derivative of a function is the gradient $\left(\frac{dy}{dx}\right)$ of the function.

Let $y = 3 + 5x - 2x^3$

$$\therefore \frac{dy}{dx} = 0 + 5 - 6x^2 = 5 - 6x^2$$

Therefore the derivative of $3 + 5x - 2x^3 = 5 - 6x^2$.

Differentiatio

The rule for differ

powers.

Worked exa

1 Differentiate
2 Find $\frac{ds}{dt}$ if $s =$

Answers

1 $y = 6\sqrt{x^3} + 2$

$y = 6x^{\frac{3}{2}} + 2x$

$\therefore \frac{dy}{dx} = \frac{3}{2} \times 6$

$= 9x^{\frac{1}{2}} +$

$= 9\sqrt{x} +$

2 $s = \frac{3}{t^2} - \frac{1}{t}$

$= 3t^{-2} - t^{-1}$

$\therefore \frac{dt}{ds} = -6t$

$= \frac{-6}{t^3} +$

Activity 2

1 Differentiate
4 a) $y = 2x$
d) $y = x^2 - 4$
g) $y = 1 - 2x$
2 Differentiate
a) $y = 8x^2$
d) $y = 2x +$
g) $v = 2 - t -$
j) $y = \frac{3}{x} - \frac{1}{x^2}$
3 Find the grad
a) $y = 2x^3 -$
c) $y = \sqrt{x} + 1$
e) $y = 2x^2 -$
g) $s = t^3 - 1,$
4 If $f(x) = 3x^3 -$
a) $f'(x)$
5 Determine th
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on.

Differentiation of fractional and negative powers

The rule for differentiating integer powers also applies to fractional and negative powers.

$(x^3)^{\frac{3}{2}}$

Worked example 4

1 Differentiate $y = 6\sqrt{x^3} + 2\sqrt{x}$ with respect to x.

2 Find $\frac{ds}{dt}$ if $s = \frac{3}{t^2} - \frac{1}{t}$.

Answers

1 $y = 6\sqrt{x^3} + 2\sqrt{x}$

$y = 6x^{\frac{3}{2}} + 2x^{\frac{1}{2}}$

$\therefore \frac{dy}{dx} = \frac{3}{2} \times 6x^{\frac{3}{2}-1} + \frac{1}{2} \times 2x^{\frac{1}{2}-1}$

$= 9x^{\frac{1}{2}} + x^{-\frac{1}{2}}$

$= 9\sqrt{x} + \frac{1}{\sqrt{x}}$

2 $s = \frac{3}{t^2} - \frac{1}{t}$

$= 3t^{-2} - t^{-1}$

$\therefore \frac{dt}{ds} = -6t^{-3} + t^{-2}$

$= \frac{-6}{t^3} + \frac{1}{t^2}$

Activity 2

1 Differentiate the following functions using the formula.

a) $y = 2x$ b) $y = 2x + 3$ c) $y = 2x^2$
d) $y = x^2 - 4x$ e) $s = 2t - t^2$ f) $s = 3t^2 + 4t - 5$
g) $y = 1 - 2x^2$ h) $y = 3x^2 + 2x - 6$ i) $y = (2x + 1)^2$

2 Differentiate the following functions with respect to the variable.

a) $y = 8x^2$ b) $y = 2 - 4x^2$ c) $y = 3x^3 - 2x + 4$
d) $y = 2x + 3$ e) $y = 3x^3 - 2x^2$ f) $s = t^2 - 3t + 4$
g) $v = 2 - t - t^2$ h) $y = x^3 - 3x^2 + 22$ i) $s = 2t^{-3} + 4t^2 - 5$
j) $y = \frac{3}{x} - \frac{1}{x^2}$ k) $y = \sqrt{x} - \frac{4}{\sqrt{x}}$ l) $y = \frac{1}{\sqrt[3]{x}} + 2\sqrt{x}$

3 Find the gradient of each of the following curves at the point indicated.

a) $y = 2x^3 - 3x$, (2, 10) b) $y = 4 - 3x - 2x^3$, (0, 4)
c) $y = \sqrt{x} + 1$, (4, 3) d) $s = 5 - 4t$, (–1, 9)
e) $y = 2x^2 - 3x + 5$, (2, 7) f) $y = 2\sqrt{x} + \frac{1}{x}$, (1, 3)
g) $s = t^3 - 1$, (1, 0) h) $s = \frac{1}{t} - t^2$, (–1, 2)

4 If $f(x) = 3x^3 - 2x^2 + 2x - 4$ calculate:

a) $f'(x)$ b) $f'(2)$ c) $f'(0)$ d) $f'(-1)$

5 Determine the gradient of the tangent to the curve $y = x^3 - 3x^2 + 2$ at the point where $x = 1$.

Activity 2 (continued)

6 Determine the gradient of the tangent to the curve $s = 2 - 3t - 4t^2$ at the point where $t = 0$.

7 Find the coordinates of the point on the curve $y = 4x - x^2$ where the gradient is zero.

8 Find the coordinates of the point on the curve $y = 4x + x^2$ where the gradient is 8.

9 Find the coordinates of the points on the curve $y = x^3 - x^2 - 6x + 2$ where the tangent is parallel to the x-axis.

The Chain, Product and Quotient Rules for differentiation

The Chain Rule (differentiation of a function of a function)

If $y = au^n$, where u is a function of x, and a and n are constants, then

$$\frac{dy}{dx} = nau^{n-1} \times \frac{du}{dx} \text{ or } \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Worked example 5

1 Calculate $\frac{dy}{dx}$ if $y = 4(2x - 3)^8$.

2 Determine the gradient of the curve $x = 2\sqrt{t^2 + 3}$ at the point where $t = 1$.

3 Find the points on the curve $y = \frac{2}{x^3 - 3x - 4}$ where the gradient is 0.

Answers

1 $y = 4(2x - 3)^8$

Let $u = 2x - 3$, so $y = 4u^8$

$\therefore \frac{du}{dx} = 2$ and $\frac{dy}{du} = 32u^7 = 32(2x - 3)^7$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 32(2x - 3)^7 \times 2$$
$$= 64(2x - 3)^7$$

2 $x = 2\sqrt{t^2 + 3}$

Let $u = t^2 + 3$, so $x = 2u^{\frac{1}{2}}$

$\therefore \frac{du}{dt} = 2t$ and $\frac{dx}{du} = u^{-\frac{1}{2}} = \frac{1}{\sqrt{u}} = \frac{1}{\sqrt{t^2 + 3}}$

$$\frac{dx}{dt} = \frac{dx}{du} \times \frac{du}{dt}$$
$$= \frac{1}{\sqrt{t^2 + 3}} \times 2t$$
$$\therefore \frac{dx}{dt} = \frac{2t}{\sqrt{t^2 + 3}}$$

At the point where $t = 1$, $\frac{dx}{dt} = \frac{2 \times 1}{\sqrt{1^2 + 3}} = 1$

So the gradient of the curve at the point where $t = 1$ is 1.

$- 3t - 4t^2$ at the

where the gradient

where the gradient

$- 6x + 2$ where the

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nt where $t = 1$.
ent is 0.

Worked example 5 (continued)

3 Write $y = \frac{2}{x^3 - 3x^2 - 4}$ as $y = 2(x^3 - 3x^2 - 4)^{-1}$

Let u $= x^3 - 3x^2 - 4$, so $y = 2u^{-1}$

$\therefore \frac{du}{dx} = 3x^2 - 6x$ and $\frac{dy}{du} = -2u^{-2} = -2(x^3 - 3x^2 - 4)^{-2}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= -2(x^3 - 3x^2 - 4)^{-2} \times (3x^2 - 6x)$$

$$\therefore \frac{dy}{dx} = \frac{-2(3x^2 - 6x)}{(x^3 - 3x^2 - 4)^2}$$

If the gradient is zero, then:

$$\frac{-2(3x^2 - 6x)}{(x^3 - 3x^2 - 4)^2} = 0$$

$$\therefore -2(3x^2 - 6x) = 0$$

$$\therefore 3x(x - 2) = 0$$

$$\therefore x = 0 \text{ or } x = 2$$

If $x = 0$, $y = \frac{2}{0^3 - 3 \times 0^2 - 4} = -\frac{1}{2}$

If $x = 2$, $y = \frac{2}{2^3 - 3 \times 2^2 - 4} = -\frac{1}{4}$

Therefore the gradient is zero at the points $(0, -\frac{1}{2})$ and $(2, -\frac{1}{4})$.

The Product Rule

If $y = uv$, where u and v are functions of x, then:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Worked example 6

Find $\frac{dy}{dx}$ if $y = x^2\sqrt{2x + 1}$

Answer

Let $u = x^2$ and $v = \sqrt{2x + 1}$

$\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = \frac{1}{2}(2x + 1)^{-\frac{1}{2}} \times 2 = \frac{1}{\sqrt{2x + 1}}$ (Use the product rule to find $\frac{dv}{dx}$ first)

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = x^2 \times \frac{1}{\sqrt{2x + 1}} + \sqrt{2x + 1} \times 2x$$

$$\therefore \frac{dy}{dx} = \frac{5x^2 + 2x}{\sqrt{2x + 1}}$$

The Quotient Rule

If $y = \frac{u}{v}$, where u and v are functions of x, then:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Worked example 7

Differentiate y if $y = \frac{3x - 5}{x^2 - 4x}$.

Answer

Let $u = 3x - 5$ and $v = x^2 - 4x$

$\therefore \frac{du}{dx} = 3$ and $\frac{dv}{dx} = 2x - 4$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\therefore \frac{dy}{dx} = \frac{(x^2 - 4x) \times 3 - (3x - 5)(2x + 4)}{(x^2 - 4x)^2}$$

$$\therefore \frac{dy}{dx} = \frac{-3x^2 - 10x + 20}{(x^2 - 4x)^2}$$

Activity 3

1 Find $\frac{dy}{dx}$.

a) $y = (2x + 9)^{10}$ b) $y = (3x - 2)^8$

c) $y = (3 - 5x)^6$ d) $y = (2x - x^2)^{-3}$

e) $y = \sqrt{9x - 3}$ f) $y = \frac{4}{7 - 2x}$

2 Find the coordinates of the point on the curve $y = (2x - 9)^4$ where the gradient is equal to 0.

3 Differentiate.

a) $x^3(x - 5)^4$ b) $(5t - 3)(t^2 + 4)^3$ c) $(x - x^2)(3 - x)^4$

d) $x\sqrt{7x - 3}$ e) $(9x - x^2)^4(2 - x)$ f) $\sqrt{5t + 1}(3t - 8)$

4 Find the coordinates on the curve $y = (x - 3)(x - 1)^2$ where the gradient is equal to 0.

5 Calculate $\frac{dy}{dx}$.

a) $y = \frac{x}{x + 9}$ b) $y = \frac{3x}{x^2 - 12}$ c) $y = \frac{5x - 7}{x + 5}$

d) $y = \frac{3 - 4x}{2x^2 + x}$ e) $y = \frac{x^2 - 5}{x^3 + 9x}$ f) $y = \frac{3x^2}{\sqrt{5x + 9}}$

6. Find the gradient of the curve $y = \frac{3x^2 - 2}{\sqrt{5x + 4}}$ at the point $(1, \frac{1}{3})$.

Calculating the equations of tangents and normals

At a point of contact, the tangent and the normal to a curve are perpendicular to each other. Fig. 7.6 shows curve $y = f(x)$ and its tangent and normal at point P.

If the gradient of the tangent is $\frac{dy}{dx}$, then the gradient of the normal is $-\frac{dx}{dy}$ or $-\frac{1}{\frac{dy}{dx}}$

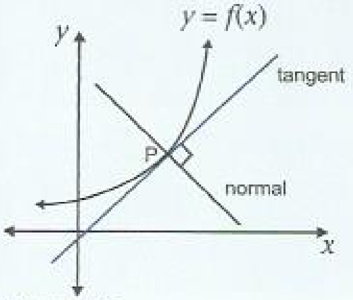

Figure 7.6

Worked exan

① Find the grad
3
at the point (

2 The diagram

$y = \frac{1}{2 - x}$. The

the curve at I

at the points

a) the equat

b) the equat

c) the area c

Answers

1 $y = x^2 + 2x +$

$\frac{dy}{dx} = 2x + 2$

At the point (

So the gradie

Gradient of t

2 a) $y = \frac{1}{2 - x}$

$\frac{dy}{dx} = \frac{1}{(2 - x}$

At the point (

$\therefore$ gradient of

Equation of t

b) Gradient

Equation

c) To work c

points Q

The coord

The coord

Length of

Length of

Area of a

Therefore

^{4} where the

c) $(x - x^2)(3 - x)^4$

f) $\sqrt{5t+1}(3t-8)$

the gradient is

c) $y = \frac{5x-7}{x+5}$

f) $y = \frac{3x^2}{\sqrt{5x+9}}$

and

$y = f(x)$

Worked example 8

3

1 Find the gradients of the tangent and the normal to the curve $y = x^2 + 2x + 2$ at the point (2, 10).

2 The diagram shows part of the curve $y = \frac{1}{2-x}$. The tangent and the normal to the curve at P(1, 1) intersect the x-axis at the points R and Q respectively. Find

- a) the equation of the tangent at P
- b) the equation of the normal at P
- c) the area of ΔPQR.

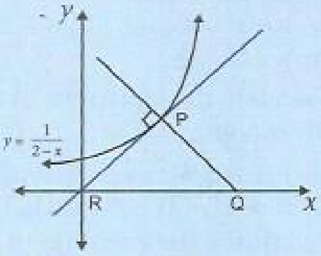

Figure 7.7

Answers

1 $y = x^2 + 2x + 2$

$\frac{dy}{dx} = 2x + 2$

At the point (2, 10), gradient $= 2 \times 2 + 2$

$= 4 + 2$

$= 6$

So the gradient of the tangent at the point (2, 10) is 6.

Gradient of the normal $= -\frac{1}{\frac{dy}{dx}} = -\frac{1}{6}$.

2 a) $y = \frac{1}{2-x}$

$\frac{dy}{dx} = \frac{1}{(2-x)^2}$

At the point (1, 1), $\frac{dy}{dx} = 1$

∴ gradient of the tangent at P = 1

Equation of the tangent is $y - 1 = 1(x - 1)$

or $y = x$

b) Gradient of the normal at P $= -\frac{1}{\frac{dy}{dx}} = -\frac{1}{1} = -1$

Equation of the normal at P is $y - 1 = -1(x - 1)$

$y = 2 - x$

c) To work out the area of ΔPQR we need to find the coordinates of the points Q and R and the lengths of PQ and PR.

The coordinates of Q are (2, 0)

The coordinates of R are (0, 0)

Length of PQ $= \sqrt{(1-0)^2 + (1-0)^2} = \sqrt{2}$ units

Length of PR $= \sqrt{(2-1)^2 + (0-1)^2} = \sqrt{2}$ units

Area of a triangle $= \frac{1}{2}bh$

$= \frac{1}{2} \times \sqrt{2} \times \sqrt{2}$

$= 1$

Therefore the area of ΔPQR = 1 square unit.

Activity 4

1 Find the gradients of the tangent and the normal to the curve at the points indicated.

a) $y = x^2 - 4x + 2$	(0, 2)	b) $y = x^3 - x$	(2, 6)
c) $y = \frac{16}{x} - 2x$	(1, 14)	d) $y = 24x - 3x^3$	(0, 0)

2 Calculate the equations of the tangent and the normal to the curve at the given point.

a) $y = x^2 - 4x + 2$	(0, 2)	b) $y = x^3 - x$,	(2, 6)
c) $y = \frac{16}{x} - 2x$	(1, 14)	d) $y = 24x - x^3$	(0, 0)

3 Determine the equation of the tangent and the normal to the curve $s = \frac{1}{t} - t^2$ at the point (–1, –2).

4 Find the equation of the tangent and the normal to the curve $y = 3x - \frac{4}{x}$ at the point where $x = 2$.

Applications of differentiation

In the real world, we apply derivatives to the study of rates of change, in particular to velocity and acceleration.

Velocity

We have learnt that displacement is defined as the distance covered in a specified direction.

Velocity is defined as the change in displacement with respect to time.

$$\text{Velocity} = \frac{\text{displacement}}{\text{time taken}}$$

If the displacement covered in t seconds is x metres, then velocity is expressed mathematically as

$$v = \frac{dx}{dt}$$

The unit of velocity is metres per second (m/s).

Acceleration

Acceleration is defined as the rate of change of velocity with respect to time.

$$\text{Acceleration} = \frac{\text{change in velocity}}{\text{time taken}}$$

If the velocity is v, then acceleration a is expressed as

$$a = \frac{dv}{dt}$$

$$\text{but } v = \frac{dx}{dt}$$

$$\text{so } a = \frac{d^2x}{dt^2}$$

So we see that acceleration is the second derivative of displacement.

The unit of acceleration is metres per second per second (m/s²).

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Worked example 9

1 An object moves in a straight line so that at time t seconds, its displacement x metres from a fixed point O is given by $x = t^2 - 3t + 2$, calculate:

- a) the initial displacement of the object
- b) the times when the object is at O
- c) the velocity of the object at time t
- d) the velocity of the object when $t = 1$.

2 The distance s metres travelled by a particle moving in a straight line in time t seconds is given by $s = t^3 - 5t^2 + 8t$, calculate:

- a) the initial velocity of the particle
- b) the times when the particle is momentarily at rest
- c) the time at which the object is moving with constant speed
- d) the time at which the acceleration is 2 m/s².

3 An object is projected vertically upwards and its height, h metres, from the point of projection at time t seconds is given by $h = 20 + 8t - 5t^2$, find:

- a) the height from which the object was projected
- b) the velocity of projection
- c) the highest point reached
- d) the acceleration of the object.

Answers

1 a) To find the initial displacement, we substitute $t = 0$ in the displacement function.

$x = t^2 - 3t + 2$

Substitute $t = 0$ into the equation.

$x = 0^2 - 3 \times 0 + 2$

$\therefore x = 2$

So the initial displacement is 2 m.

b) If the object is at O, then $x = 0$, since displacement is measured from O. Substituting $x = 0$ in the displacement function we get:

$t^2 - 3t + 2 = 0$

$(t - 1)(t - 2) = 0$

$\therefore t = 1$ or $t = 2$

So the object is at O after 1 s or 2 s.

c) $x = t^2 - 3t + 2$

$v = \frac{dx}{dt} = 2t - 3$

So the velocity at time t is $2t - 3$.

d) $v = 2t - 3$, if $t = 1$

$v = 2 \times 1 - 3 = -1$

So the velocity when $t = 1$ is -1 m/s.

Worked example 9 (continued)

2 a) $s = t^3 - 5t^2 + 8t$

$v = \frac{ds}{dt} = 3t^2 - 10t + 8$

To find the initial velocity we substitute $t = 0$ in the velocity function.

$\therefore v = 3 \times 0^2 - 10 \times 0 + 8 = 8$

So the initial velocity is 8 m/s.

b) The particle is momentarily at rest if the velocity is zero.

$\therefore 3t^2 - 10t + 8 = 0$

$(3t - 4)(t - 2) = 0$

$\therefore \quad t = 1\frac{1}{3}$ or $t = 2$

c) If the particle is moving with constant speed then the acceleration is zero.

$v = 3t^2 - 10t - 8$

$a = \frac{dv}{dt} = 6t - 10$

Substituting $a = 0$ we get:

$6t - 10 = 0$

$\therefore t = 1\frac{2}{3}$ s

The particle is moving with constant speed at $t = 1\frac{2}{3}$ s.

d) $a = 6t - 10$, substituting $a = 2$ we get:

$2 = 6t - 10$

$6t = 12$

$\therefore t = 2$

The acceleration is 2 m/s² at time 2 seconds.

3 a) $h = 20 + 8t - 5t^2$

To find the point of projection from the ground, substitute $t = 0$ in the function h:

$\therefore h = 20 + 8 \times 0 - 5 \times 0^2$

$= 20$

So the object was projected at a height of 20 m above ground level.

b) $v = \frac{dh}{dt}$

$= 8 - 10t$

At the time of projection $t = 0$

$\therefore v = 8 - 10 \times 0 = 8$

The velocity of projection is therefore 8 m/s.

c) At the highest point reached $v = 0$ so we substitute $v = 0$ into the velocity function:

$\therefore 8 - 10t = 0$

$\therefore t = \frac{4}{5}$

The object reaches its highest point at $\frac{4}{5}$ s.

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Activity 5

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Worked example 9 (continued)

If $t = \frac{4}{5}$, $h = 20 + 8 \times (\frac{4}{5}) - 5 \times (\frac{4}{5})^2 = 20 + 6\frac{2}{5} - 3\frac{1}{5} = 23.2$

So the highest point reached is 23.2 m from the point of projection.

d) $a = \frac{dv}{dt} = -10$

So the acceleration is constant at -10 m/s^2.

Activity 5

1 Given the function $x = t^2 - 5t + 6$, where x is displacement in metres and t is time in seconds, calculate:
- a) the times for which $x = 0$
- b) $\frac{dx}{dt}$ and state what it represents
- c) the time when $\frac{dx}{dt} = 0$.

2 The distance, x metres, travelled by a particle moving in a straight line in time t seconds is given by $x = 2t^2 - 3t - 4$.

Calculate:
- a) the distance travelled in 1 seconds
- b) the distance travelled in 2 seconds
- c) the distance travelled in the 2nd second.

3 A particle moves in a straight line so that at time t seconds, its displacement x metres from a fixed point O is given by $x = 3t^2 + 2t - 5$.

Calculate:
- a) the initial displacement of the particle
- b) the times when the particle is at O
- c) the velocity of the particle at time t
- d) the velocity of the particle when $t = 1$.

4 The displacement s metres travelled by a particle moving in a straight line in time t seconds is given by $s = t^3 - 5t^2 + 8t$.

Calculate:
- a) the initial velocity of the particle
- b) the times when the particle is momentarily at rest
- c) the time at which the object is moving with constant speed
- d) the time when the acceleration is 2 m/s^2.

5 A particle is projected vertically upwards and its height, h metres, from the ground after time t seconds is given by $h = 10 + 8t - 5t^2$.
- a) Explain how you know that the particle was not projected from ground level.
- b) Find:
  - i) the height of the particle after 2 seconds
  - ii) the velocity of the particle after 1 second
- c) Show that the acceleration of the particle is constant.

SUB-TOPIC 2 Integration

Integration in Mathematics is often described as the reverse process of differentiation. We have already seen how integration is used to find areas under curves.

The antiderivative or indefinite integral

You have learnt that if $y = x^2$, then the derivative $\frac{dy}{dx} = 2x$. Similarly, the derivatives of $x^2 + 4$ and $x^2 + 10$ equal $2x$ as well.

So if we want to know what equation $2x$ is the derivative of, we find that there are a number of equations that satisfy this requirement, so we need to represent the answer by adding a constant C, to x^2.

We say that $x^2 + C$ is the **antiderivative or indefinite integral** of $2x$. Integration is the reverse process of differentiation.

So if $\frac{dy}{dx} = f(x)$ defines the gradient (derivative) of a curve at a point, we **integrate** $f(x)$ with respect to x in order to find the equation of the curve whose gradient is $f(x)$.

The symbol $\int$ means "the integral of", so $\int f(x)dx$ means the integral of $f(x)$ with respect to x.

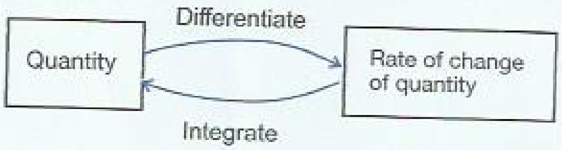

Figure 7.8

New words

integral: the area under the graph of a function; it is found by calculating the antiderivative

integration: the process of finding the integral

Explanation of the integration notation

Consider the integral for distance $\int v(t)dt$

This sign means that we need to integrate

$$\int v(t)dt$$

This represents "delta t" which means "change in time". It shows that we are looking at a particular time interval.

(It does not mean $d \times t$.)

This indicates the area under the graph of $v(t)$ which we need to integrate

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2 Integrate

a) $\int(x^2 -$

3 Find the

$\frac{dy}{dx} = 2x^3 +$

Answers

1 a) $\int 3x^2 dx$

b) $\int(3x -$

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Calculating the antiderivative (indefinite integral)

Consider this example where the derivative of $x^2 + 3x + 10 = 2x + 3$.

Similarly, the derivatives of $x^2 + 3x$ and $x^2 + 3x + 30$ and $x^2 + 3x - 4$ are also equal to $2x + 3$.

To find the antiderivative we need to reverse the process:

$$\int(2x + 3)dx = \left(\frac{2x^{1+1}}{1+1} + \frac{3x^{0+1}}{0+1}\right) + C \qquad [\text{note that } 3 = 3x^0]$$
$$= x^2 + 3x + C$$

It's a good idea to check your answer by differentiating the antiderivative to get back to the derivative.

The rule for calculating the antiderivative is:
$\int ax^n dx = \frac{ax^{n+1}}{n+1} + C$ where a, n and C are constants and $n \neq -1$.

The variable C is called the constant of integration, or the arbitrary constant. We cannot find the value of C from the formula alone – it could take on some different values. We will learn how we can do this later in this topic.

New word

arbitrary constant: a symbol that can have various values, but which is not affected by changes in the values of the variables of the equation.

Worked example 10

1 Calculate the antiderivative.

a) $\int 3x^2 dx$ b) $\int(3x - 4)dx$

2 Integrate.

a) $\int(x^2 - 4x + 5)dx$ b) $\int \frac{x^4 + 2x^3 - 3x}{x}dx$

3 Find the equation, in general terms, of the curve whose gradient is given by: $\frac{dy}{dx} = 2x^3 + 3x^2 - 4x$.

Answers

1 a) $\int 3x^2 dx = \frac{3x^{2+1}}{2+1} + C$
$= \frac{3x^3}{3} + C$
$= x^3 + C$

b) $\int(3x - 4)dx = \int 3x dx - \int 4x^0 dx$ (Note that 4 is written as $4x^0$)
$= \frac{3x^{1+1}}{1+1} - \frac{4x^{0+1}}{0+1} + C$
$= \frac{3x^2}{2} - 4x + C$

Note: the process of adding 1 to the power and dividing by the result is usually done mentally to avoid multiplicity.

Worked example 10 (continued)

2 a) $\int(x^2 - 4x + 5)dx = \frac{x^3}{3} - \frac{4x^2}{2} + 5x + C$

$= \frac{x^3}{3} - 2x^2 + 5x + C$

b) $\int\frac{x^4 + 2x^2 - 3x}{x}dx = \int(x^3 + 2x - 3)dx$ Divide each term by x.

$= \frac{x^4}{4} + \frac{2x^2}{2} - 3x + C$

$= \frac{x^4}{4} + x^2 - 3x + C$

3 $\frac{dy}{dx} = 2x^3 + 3x^2 - 4x$ Integrate both sides of the equation with respect to x.

$\int\frac{dy}{dx}dx = \int(2x^3 + 3x^2 - 4x)dx$

$\int dy = \int(2x^3 + 3x^2 - 4x)dx$

$y = \frac{2x^4}{4} + \frac{3x^3}{3} - \frac{4x^2}{2} + C$

$y = \frac{x^4}{2} + x^3 - 2x + C$

Integration of functions with fractional or negative powers

The same formula is used to find indefinite integrals of expressions with fractional powers or negative powers.

Worked example 11

1 Integrate $\int(\sqrt{x} - \frac{2}{x^2})dx$

2 Integrate $\int(\frac{4}{\sqrt[3]{x}} - \frac{2}{\sqrt{x}} + 3\sqrt{x^3})dx$

Answers

1 $\int(\sqrt{x} - \frac{2}{x^2})dx = \int(x^{\frac{1}{2}} - 2x^{-2})dx$

$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{2x^{-1}}{-1} + C$

$= \frac{2\sqrt{x^3}}{3} + \frac{2}{x} + C$

2 $\int(\frac{4}{\sqrt[3]{x^2}} - \frac{2}{\sqrt{x}} + 3\sqrt{x^3})dx = \int(4x^{-\frac{2}{3}} - 2x^{-\frac{1}{2}} + 3x^{\frac{3}{2}})dx$

$= \frac{4x^{\frac{1}{3}}}{\frac{1}{3}} - \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + \frac{3x^{\frac{5}{2}}}{\frac{5}{2}} + C$

$= 12\sqrt[3]{x} - 4\sqrt{x} + \frac{6\sqrt{x^5}}{5} + C$

Activity 6

1 Integrate.

a) $\int(x^{\frac{1}{4}} - \frac{1}{x^4})dx$

b) $\int(x - \frac{4}{\sqrt[4]{x}} + 3\sqrt{x^5})dx$

c) $\int(3x^{-\frac{1}{2}} + \frac{1}{3x^2})dx$

Definite integrals

$\int f(x)dx$ is an indefinite integral, i.e. the limits of integration are not given. We insert the constant of integration because the limits are not known.

$\int_a^b f(x)dx$ is a definite integral; the function $f(x)$ is to be integrated for values of x from a to b, where a is the lower limit and b the upper limit.

Theorem: $\int_a^b f(x)dx = [F(x) + C]_a^b$
$= F(b) - F(a)$

Note

- $F(x)$ is the integral of $f(x)$.
- In a definite integral the constant of integration disappears.

Worked example 12

1 Find the value of $\int_1^2 x^2dx$.

2 Evaluate $\int_{-1}^{2}(2x - 3)dx$.

3 Evaluate $\int_4^9\left(2\sqrt{t} - \frac{1}{\sqrt{t}}\right)dt$

Answers

1 $\int_1^2 x^2dx = \left[\frac{x^3}{3}\right]_1^2$

$= \left[\frac{2^3}{3} - \frac{1^3}{3}\right]$

$= \frac{8}{3} - \frac{1}{3}$

$= 2\frac{1}{3}$

2 $\int_{-1}^{2}(2x - 3)dx = [x^2 - 3x]_{-1}^{2}$

$= [(2)^2 - 3(2) - ((-1)^2 - 3(-1))]$

$= -6$

3 $\int_4^9\left(2\sqrt{t} - \frac{1}{\sqrt{t}}\right)dt = \int_4^9 (2t^{\frac{1}{2}} - t^{-\frac{1}{2}})dt$

$= \left[\frac{4t^{\frac{3}{2}}}{3} - 2t^{\frac{1}{2}}\right]_4^9$

$= \left[(\frac{4 \times 9^{\frac{3}{2}}}{3} - 2 \times 9^{\frac{1}{2}}) - (\frac{4 \times 4^{\frac{3}{2}}}{3} - 2 \times 4^{\frac{1}{2}})\right]$

$= (36 - 6) - (\frac{32}{3} - 4)$

$= 23\frac{1}{3}$

Activity 7

1 Integrate these linear functions.

a) $\int(x + 5)dx$ b) $\int(4x + 5)dx$ c) $\int(6x - 3)dx$

d) $\int(3 - 8x)dx$ e) $\int(1 - x)dx$ f) $\int 9dx$

2 Integrate these quadratic functions.

a) $\int(2x^2 + 8x)dx$ b) $\int(10x^2 + 6x - 3)dx$

c) $\int(6x^2 - 2x + 7)dx$ d) $\int(5 - 6x - 3x^2)dx$

e) $\int\frac{8x^5 - 4x}{x}dx$ f) $\int(6\sqrt{x} - \frac{2}{\sqrt{x}} + 4)dx$

g) $\int(12x^5 - 2x^3 - 8)dx$ h) $\int\frac{7x^7 + 4x^4 - 2}{x}dx$

3 Evaluate the integrals between the given x-values.

a) $\int_0^1 xdx$ b) $\int_0^2 x^2dx$ c) $\int_1^2(x - 1)dx$

d) $\int_{-1}^1(3x^2 + 7)dx$ e) $\int_1^2(2x - 6x^2)dx$ f) $\int_{-1}^0(2x - 6x^2)dx$

g) $\int_{-1}^2(6x^2 + 8x)dx$ h) $\int_0^3(4x - 9x^2)dx$ i) $\int_1^2(3x^2 + 2x - 5)dx$

4 Given that $\int(9x^2 - 4x + 10)dx = ax^3 + bx^2 + cx$, where a is a constant, find the integers a, b and c.

5 A curve is defined by $\frac{dy}{dx} = 3 + 4x - 12x^2$. Given that the curve passes through the point (1, 2), find the equation of the curve.

Area under a graph

In Topic 3 you saw how to approximate the area under a curve.

The area bounded by the curve $y = f(x)$, the x-axis and the values $x = a$ and $x = b$, is given by $A = \int_a^b ydx$

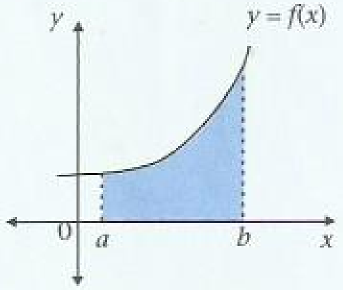

Figure 7.9

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Answers

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Figure 7.11

$A = \int_a^b ydx$

$= \int_0^2(x^2 + 1$

$= \left[\frac{x^3}{3} + x\right.$

$= \left[(\frac{8}{3} + 2\right.$

$= 4\frac{2}{3}$ unit

2 a) At the

$\therefore x^2 - 4$

$\therefore x(x -$

$\therefore x = 0$

So the

b) $A = \int_a^b y$

$= \int_0^4(x$

$= \left[\frac{x^3}{3}\right.$

$= (\frac{4^3}{3}$

$= -10$

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Note: T

$\int(6x - 3)dx$

$\int 9dx$

$\int(x - 1)dx$

$\int(2x - 6x^2)dx$

$\int(3x^2 + 2x - 5)dx$

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Worked example 13

1 Find the area bounded by the curve $y = x^2 + 1$, the x-axis and lying between $x = 0$ and $x = 2$.

2 The diagram shows part of the curve $y = x^2 - 4x$. Find

a) the coordinates of the points A and B

b) the area of the shaded region.

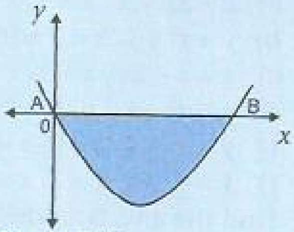

Figure 7.10

Answers

1 The curve $y = x^2 + 1$ is as shown below.

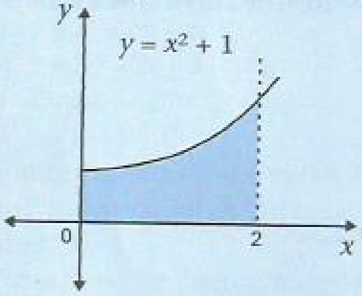

Figure 7.11

$$A = \int_a^b y dx$$

$$= \int_0^2 (x^2 + 1)dx$$

$$= \left[\frac{x^3}{3} + x\right]_0^2$$

$$= \left[(\tfrac{8}{3} + 2) - (0 + 0)\right]$$

$$= 4\tfrac{2}{3} \text{ units}^2$$

2 a) At the points A and B, $y = 0$

$\therefore x^2 - 4x = 0$

$\therefore x(x - 4) = 0$

$\therefore x = 0$ or $x = 4$

So the coordinates of A and B are (0, 0) and (4, 0).

b) $A = \int_a^b y dx$

$$= \int_0^4 (x^2 - 4x)dx$$

$$= \left[\frac{x^3}{3} - 2x^2\right]_0^4$$

$$= (\tfrac{4^3}{3} - 2 \times 4^2) - (0 - 0)$$

$$= -10\tfrac{2}{3}$$

The area is $10\frac{2}{3}$ square units.

Note: The negative sign shows that the area lies below the x-axis.

Activity 8

1 Find the area bounded by the curve and the given x-values and the x-axis.
   - a) $y = 2x + 3$, $x = 1$, $x = 2$
   - b) $y = x^2 + 2$, $x = 2$ and $x = 3$
   - c) $y = x - 2x^2$, $x = \frac{1}{2}$, $x = 1$
   - d) $y = x^2 - x - 6$, $x = 3$ and $x = 4$
   - e) $y = x^2 - 4$, $x = 0$, $x = 2$
   - f) $y = \frac{2}{x^2} + x$, $x = 2$, $x = 3$

2 Find the area bounded by the curve $y = 3x^2 - 2x - 5$ and the x-axis.

3 The diagram shows part of the curve $y = 2x^2 - 18$. The curve crosses the x-axis at A and B, find:
   - a) the coordinates of the points A and B
   - b) the area of the shaded region.

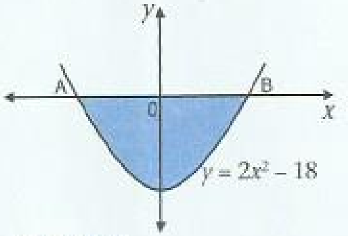

Figure 7.12

4 The area shown shaded in the diagram is 26 square units. Find the value of a.

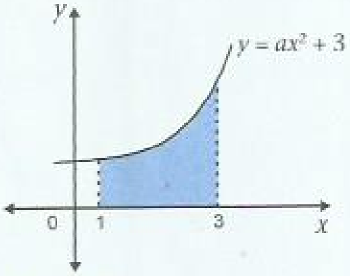

Figure 7.13

Revision

1 Calculate.
   - a) $\lim_{x \to 3} \frac{2x + 2}{x - 2}$
   - b) $\lim_{x \to 4} \frac{x^2 - 4}{x + 2}$
   - c) $\lim_{x \to 3} \frac{x^2 - 4x}{x - 3}$

2 Differentiate
   - a) $y = x^2 - 2x$
   - b) $v = 2t^2 - 2$
   - c) $s = 5 - 3t -$
   - d) $y = 2 - x -$

3 Find the poin

4 Find the equa
point (2, –3).

5 The distance,
t seconds is gi
   - a) the distan
   - b) the distan
   - c) the distan

6 Find the area

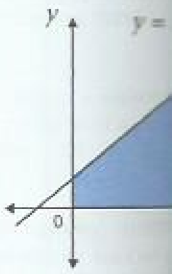

Figure 7.14a

7 Calculate.
   - a) $\int(1 - x - 6$
   - b) $\int(x^2 + x - 7$
   - c) $\int(4\sqrt{x} - \frac{2}{x^2})$
   - d) $\int(4x^3 - 2x)$

es and the x-axis.

he x-axis.
ve crosses the

Find the value

TOPIC
7

Summary, revision and assessment

Revision

1 Calculate. ✓

a) $\lim\limits_{x \to 3} \frac{2x+2}{x-2}$ ✓

b) $\lim\limits_{x \to 4} \frac{x^2-4}{x+2}$ ✓

c) $\lim\limits_{x \to 3} \frac{x^2-4x+3}{x-3}$

2 Differentiate by first principles.

a) $y = x^2 - 2x - 4$

b) $v = 2t^2 - 2t + 3$

c) $s = 5 - 3t - 2t^2$

d) $y = 2 - x - x^2$

3 Find the points on the curve $y = x^3 - x^2 - 5x + 4$, where the gradient is zero.

4 Find the equation of the tangent and the normal to the curve $y = \frac{2}{x} - x^2$ at the point (2, –3).

5 The distance, x metres, travelled by a particle moving in a straight line in time t seconds is given by $x = t^3 + 2t^2 - 3t$, find:

a) the distance travelled in two seconds

b) the distance travelled in three seconds

c) the distance travelled in the third second.

6 Find the area of the shaded regions in Fig. 7.14 .

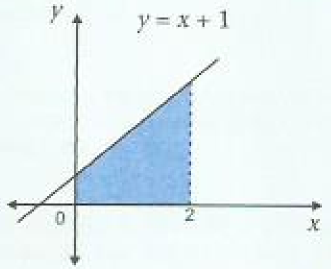

Figure 7.14a

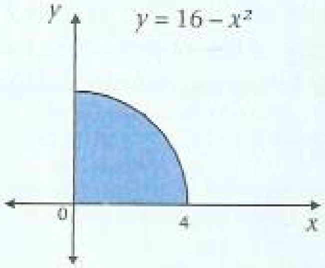

Figure 7.14b

7 Calculate.

a) $\int(1 - x - 6x^2)dx$

b) $\int(x^2 + x - 7)dx$

c) $\int(4\sqrt{x} - \frac{2}{x^2})dx$

d) $\int(4x^3 - 2x)dx$

Summary, revision and assessment (continued)

Assessment

✓ 1 Differentiate by first principles:
  a) $f(x) = -4x^2$ ✓
  b) $y = x^3 + 3x^2 - 8x + 4$

2 a) Calculate $\frac{dy}{dx}$ if $y = \frac{3}{2x} - \frac{x^2}{2}$.
  b) Calculate $f'(1)$ if $f(x) = (7x + 1)^2$.

3 a) Find the gradient of the tangent to the curve $f(x) = x^3 - 2x^2 + 6$ at the point where $x = 1$.
  b) Find the coordinates of the point on the curve $f(x) = 4x + 2x^2$ where the gradient is 4.
  c) Find the coordinates of the points on the curve $f(x) = -2x^3 - 3x^2 - 4x + 5$ where the tangent is parallel to the x-axis.

4 If $f(x) = x^3 - 3x^2 + 2x - 5$, find:
  a) $f'(x)$
  b) $f'(-1)$
  c) $f'(3)$

5 An object moves in a straight line so that at time t seconds, its displacement x metres from a fixed point O is given by $x = t^2 - 5t + 6$. Find:
  a) the initial displacement of the object
  b) the times when the object is at O
  c) the velocity of the object at time t
  d) the velocity of the object when $t = 1$.

6 Calculate the following definite integrals:
  a) $\int_1^4 \left(3\sqrt{t} + \frac{1}{\sqrt{t}}\right) dt$
  b) $\int_{-1}^{2} \frac{5x^6 + 4x^5 - 3}{x^2} dx$
  c) $\int_1^9 \left(\frac{1}{\sqrt{x}} + \sqrt{x}\right) dx$

7 The area bounded by the line $y = 2x + 3$ and between $x = a$ and $x = 3$ is 20 units^2. Find the value of a.

8 If $\frac{dy}{dx} = 2 - x - x^2$, find y in terms of x, given that when $x = 0$, $y = 2$.

Glossary

A

arbitrary constant
various values, b
changes in the va
equation. 189
asymptote: A line
never reaches. 2

C

Cartesian plane (c
containing the x-
collinear: lying on
congruent/isometr
transformation th
and dimensions o
constraints: the co
in an optimisation
cubic function: an
of 3. 7

D

derivative: the deriv
equal to the gradie
differentiate: to fin
function 3

E

elimination: solving
eliminating (gettin
by doing operation
combining them.

F

free vector: a vector
in position of a po
plane. 85

H

head: terminal (end
hemisphere: half of
has been sliced in h
middle) 150

I

integral: the area un
is found by calculat
integration: the proc
integral 188

K

knot (kn): unit of spe
mile per hour 163

L

logarithm: The logari
exponent to which
produce the numbe

Mathematics LEARNER'S BOOK **12**

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